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Derivation of inflection points of nonlinear regression curves - implications to statistics

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Abstract: In this paper, we derive inflection points for the commonly known growth curves, namely, generalized logistic, Richards, Von Bertalanffy, Brody, logistic, Gompertz, generalized Weibull, Weibull, Monomolecular and Mitscherlich functions. The functions often represent the mean part of non-linear regression models in Statistics. Inflection point of a growth curve is the point on the curve at which the rate of growth gets maximum value and it represents an important physical interpretation in the respective application area. Not only the model parameters but also the inflection point of a growth curve is of high statistical interests.

Keywords: Inflection Point, Growth Model, Gompertz, Logistic, Richards, Weibull

1. Introduction

The purpose of this paper is to explore the existence of inflection points for the commonly known growth functions. These functions are often representing the mean part of non-linear regression models in Statistics. For example, [1-2] define the non-linear growth curve as mean part of the statistical model as:

$$Y_t = f(t) + \varepsilon_t \tag{1}$$

where Y_t is observation on growth, $f(t)$ is mean growth function, and ε_t is random error at time t. The model parameters are estimated from observations.

The Mathematical definition of inflection point as given in [3] is: Inflection point of a continuous function $f(t)$ is a point $t = a$, on an open interval containing point $t = a$, where the second derivative $f''(t) < 0$ on one side and $f''(t) > 0$ on the other side of $t = a$, and $f''(a)$ is either 0 or does not exist.

In practice, inflection point is the point at which the rate of growth gets maximum value. There are some interesting applications and practical uses of inflection points in areas including economics, computer science, demography, animal science, plant science, forestry and biology.

In economic growth, the relationship between inflation and long-run economic growth is negative and a nonlinear one [4]. In [5] it is reported that if such a nonlinear relationship exists, then it should be possible, in principle, to estimate the inflection point, or a threshold, at which the sign of the relationship between the two variables would switch. Even in computer sciences, inflection point is important in software reliability growth modeling [6-8]. In demography, ageing populations is of concern and growth models are used to analyze the impact of ageing populations on public spending. The inflection point divides the period in the curve [9]. Furthermore, inflection point is useful in analyzing mortgage as done in Canada [10].

In biological, growth models have been widely used in many areas especially animal, plant and forestry sciences. Applications in animal sciences include [11-17]; and those in forestry and plant sciences include [18-20].

It has been shown by [21] that the commonly known growth models are solutions of the rate-state first order ordinary differential equation. Researchers [22] introduce a new generalized growth model. In latter paper, inflection points are discussed. In the present paper, we aim to derive inflection point of commonly known growth models and emphasis their roles in Statistics. We show that inflection point is expressed as a function of model parameters.

In Section 2, we define inflection point of a function $f(t)$ and express the procedure to derive it. We present detailed derivations of inflection points. List of the inflection points of the growth functions are also provided in Table 1. In Section 3, conclusions are given.

2. Derivation of Inflection Points

2.1. Generalized Logistic Function

The Generalized Logistic function as given in [23] is expressed in its original notations as $Y(t) = \mathcal{A} + \frac{\mathcal{K}-\mathcal{A}}{[1+Qe^{-\mathcal{B}(t-M)}]^{\frac{1}{\omega}}}$. This we now re-express with same notations used in this paper as $f(t) = A_L + (A - A_L)\left[1 - Be^{-k(t-\mu)}\right]^m$ where $B = \left[1 - \left(\frac{A_\mu - A_L}{A-A_L}\right)^{\frac{1}{m}}\right]$ by replacing in the original equation using the transformations as $Y(t) = f(t),\ \mathcal{A} = A_L,\ \mathcal{K} = A,\ \mathcal{B} = k,\ M = \mu$, $\omega = -\frac{1}{m}$ and $-Q = 1 - \left(\frac{A_\mu - A_L}{A-A_L}\right)^{\frac{1}{m}}$.

Inflection Point For Generalized Logistic, $f'(t)$ and $f''(t)$ are given respectively by

$$f'(t) = mk\ \ [f(t) - A_L]^{1-\frac{1}{m}}\ \left\{[A - A_L]^{\frac{1}{m}} - [f(t) - A_L]^{\frac{1}{m}}\right\}$$

$$f''(t) = mk\left\{\left(1 - \frac{1}{m}\right)\left[\frac{A - A_L}{f(t) - A_L}\right]^{\frac{1}{m}} - 1\right\}\ f'(t)$$

Now, $f''(t) = 0 \quad \Leftrightarrow \left\{\left(1 - \frac{1}{m}\right)\left[\frac{A-A_L}{f(t)-A_L}\right]^{\frac{1}{m}} - 1\right\} = 0$
$\Leftrightarrow\ f(t) = A_L + (A - A_L)\ \left[1 - \frac{1}{m}\right]^m$

$$a = \mu + \frac{1}{k}\log\left\{m\left[1 - \left(\frac{A_\mu - A_L}{A-A_L}\right)^{\frac{1}{m}}\right]\right\} \tag{2}$$

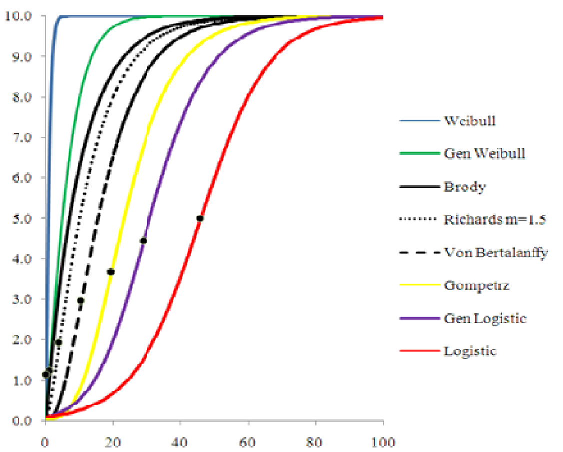

Figure 1 *Plots of the Growth Curves (lines) with their respective Inflection Points (dots)*

Equation (2) is the point of inflection since $f''(a) = 0$ at that point and also $f''(t) < 0$ and $f''(t) > 0$ are satisfied on the increasing and decreasing sides of that point. Hence, for Generalized Logistic curve, the single point of inflection occurs when the growth reaches the weight $f(a) = A_L + (A - A_L)\ \left[1 - \frac{1}{m}\right]^m$ at time $t = a$ given by (2). For $0 < m \leq 1$, we have $\log\left\{m\left[1 - \left(\frac{A_\mu - A_L}{A-A_L}\right)^{\frac{1}{m}}\right]\right\} < 0$ which indicates that the time of inflection should be less than μ. However, such inflection point does not exist, as the growth under logistic curve meets inflection point only at later age / time $a = \mu$.

The inflection points are indicated by dots on the graphs in Figure 1. It seems that there is some pattern among the inflection points, and these can have real meanings in the application areas.

2.2. Particular Case of the Generalized Logistic Function

The Particular Case of Logistic function is defined [23] as $Y(t) = \frac{\mathcal{K}}{[1+Qe^{-\alpha\nu(t-t_0)}]^{\frac{1}{\omega}}}$. This we now rewrite with same notations used in this paper as $f(t) = A\left[1 - Be^{-k(t-\mu)}\right]^m$, $B = 1 - \left(\frac{A_\mu}{A}\right)^{\frac{1}{m}}$ by replacing in the original equation using the transformations $Y(t) = f(t),\ \mathcal{K} = A,\ k = \alpha\omega,\ t_0 = \mu$, $\omega = -\frac{1}{m}$ and $(-Q) = 1 - \left(\frac{A_\mu}{A}\right)^{\frac{1}{m}}$.

Inflection Point
For Particular case of the Generalized Logistic, $f'(t)$ and $f''(t)$ are given respectively by
$f'(t) = mk\ \left\{A^{\frac{1}{m}} - f^{\frac{1}{m}}(t)\right\}\ f^{1-\frac{1}{m}}(t)$ and $f''(t) = k\left\{(m-1)\left(\frac{A}{f(t)}\right)^{\frac{1}{m}} - m\right\}\ f'(t)$. Now, $f''(t) = 0$ $\Leftrightarrow (m-1)\left(\frac{A}{f(t)}\right)^{\frac{1}{m}} = m \Leftrightarrow\ f(t) = A\ \left[1 - \frac{1}{m}\right]^m$ and this implies

$$a = \mu + \frac{1}{k}\log\left\{m\left[1 - \left(\frac{A_\mu}{A}\right)^{\frac{1}{m}}\right]\right\} \tag{3}$$

Equation (3) is the single point of inflection occurs and the respective growth is $f(a) = A\ \left[1 - \frac{1}{m}\right]^m$.

2.3. Richards Function

The Richards function is defined as in the usual notations [24] as $f(t) = A\,(1 - B\,e^{-kt})^m$, $B = 1 - \left(\frac{A_0}{A}\right)^{\frac{1}{m}}$. Here the parameter m can assume any non-zero real number.

Inflection Point
For Richards function $f'(t)$ and $f''(t)$ are given by $f'(t) = mkf^{1-\frac{1}{m}}(t)\left[A^{\frac{1}{m}} - f^{\frac{1}{m}}(t)\right]$ and $f''(t) = k\,f'(t)\,f^{-\frac{1}{m}}(t)\left[(m-1)A^{\frac{1}{m}} - mf^{\frac{1}{m}}(t)\right]$. Here, we observe that

$$f''(t) = 0 \Leftrightarrow \left[(m-1)\,A^{\frac{1}{m}} - m\,f^{\frac{1}{m}}(t)\right] = 0\ \Leftrightarrow\ f(t) = A\left(\frac{m-1}{m}\right)^m$$

and this implies

$$a = \left(\frac{1}{k}\right)\ \log\left\{m\left[1 - \left(\frac{A_0}{A}\right)^{\frac{1}{m}}\right]\right\} \tag{4}$$

Equation (4) for $m \neq 1$ is the point of inflection since $f''(a) = 0$ at that point and also $f''(t) < 0$ and $f''(t) > 0$ are satisfied on the increasing and decreasing sides of that point.

Hence, for Richards curve, the single point of inflection occurs when the growth reaches $\left(\frac{m-1}{m}\right)^m$ of its final weight, i.e. $f(a) = \left(\frac{m-1}{m}\right)^m A$ at time given by (4).

2.4. Von Bertalanffy Function

Von Bertalanffy growth function is defined [25] as $f(t) = A\,(1 - B\,e^{-kt})^3, B = 1 - \left(\frac{A_0}{A}\right)^{\frac{1}{3}}$.

Inflection Point

For Von Bertalanffy function $f'(t)$ and $f''(t)$ are respectively given by $f'(t) = 3k\, f^{\frac{2}{3}}(t)\ \left[A^{\frac{1}{3}} - f^{\frac{1}{3}}(t)\right]$ and $f''(t) = 3k^2\, f^{\frac{1}{3}}(t)\ \left[A^{\frac{1}{3}} - f^{\frac{1}{3}}(t)\right]\ \left[2\,A^{\frac{1}{3}} - 3\,f^{\frac{1}{3}}(t)\right]$. Here, we observe that $f''(t) = 0 \iff \left[2\,A^{\frac{1}{3}} - 3\,f^{\frac{1}{3}}(t)\right] = 0 \iff f(t) = (8/27)A$, and this implies

$$a = \left(\frac{1}{k}\right) \log\left\{3\left[1 - \left(\frac{A_0}{A}\right)^{\frac{1}{3}}\right]\right\} \tag{5}$$

This is the point of inflection point. Hence, for Von Bertalanffy curve, the single point of inflection occurs when the growth reaches $f(a) = (8/27)A$.

2.5. Brody Function

Brody growth function is defined [15] as $f(t) = A\,(1 - B\,e^{-kt}),\ B = 1 - \frac{A_0}{A}$.

Inflection Point

For Brody growth function $f'(t)$ and $f''(t)$ are respectively given by $f'(t) = k[A - f(t)]$ and $f''(t) = k^2[A - f(t)]$. Here, we observe that (i) $f''(t) = 0 \iff [A - f(t)] = 0 \iff f(t) = A \iff t = \infty$. That is, inflection point may occur when $t = \infty$. But, (ii) $f''(t) < 0$ for $t < \infty$ and (iii) $f''(t) > 0$ does not hold for any value of t. Hence, the Brody growth function does not possess any point of inflection since $f''(t) > 0$ is not satisfied for any value of t.

2.6. Logistic Function

The classical Logistic function [26] is defined as $f(t) = \frac{A}{1+Be^{-kt}}$, where $B = \left(\frac{A}{A_0} - 1\right)$.

Inflection Point

For Logistic function, $f'(t)$ and $f''(t)$ are given respectively by $f'(t) = k\,f(t)\left[1 - \frac{f(t)}{A}\right]$ and $f''(t) = kf'(t)\left[1 - \frac{2f(t)}{A}\right]$. Here, we observe that $f''(t) = 0 \iff \left[1 - \frac{2f(t)}{A}\right] = 0 \iff f(t) = \frac{A}{2}$ and this implies:

$$a = \left(\frac{1}{k}\right) \log\left(\frac{A}{A_0} - 1\right) \tag{6}$$

Equation (6) is the point of inflection, since $f''(a) = 0$ at that point and also $f''(t) < 0$ and $f''(t) > 0$ are satisfied on the increasing and decreasing sides of that point. Hence, for Logistic curve, the single point of inflection occurs when the growth reaches half of its final growth $f(a) = \frac{A}{2}$ at time given by (6).

Table 1 *List of Growth Functions and their respective Inflection Points Derived.*

Model name	Growth function $f(t)$	Inflection point	Growth at inflection point
Generalized Logistic	$f(t) = A_L + (A - A_L).\left[1 - Be^{-k(t-\mu)}\right]^m$, $m < 0$	$a = \mu + \frac{1}{k}\,log\left\{m\left[1 - \left(\frac{A_\mu - A_L}{A - A_L}\right)^{\frac{1}{m}}\right]\right\}$	$f(a) = A_L + (A - A_L)\left[1 - \frac{1}{m}\right]^m$
Particular Case of Logistic	$f(t) = A\left[1 - Be^{-k(t-\mu)}\right]^m$, $m < 0$	$a = \mu + \frac{1}{k}\,log\left\{m\left[1 - \left(\frac{A_\mu}{A}\right)^{\frac{1}{m}}\right]\right\}$	$f(a) = A\left[1 - \frac{1}{m}\right]^m$
Richards	$f(t) = A\,(1 - B\,e^{-kt})^m$	$a = \frac{1}{k}\,log\left\{m\left[1 - \left(\frac{A_0}{A}\right)^{\frac{1}{m}}\right]\right\}$	$f(a) = A\left[1 - \frac{1}{m}\right]^m$
Von Bertalanffy	$f(t) = A\,(1 - B\,e^{-kt})^3$	$a = \frac{1}{k}\,log\left\{3\left[1 - \left(\frac{A_0}{A}\right)^{\frac{1}{3}}\right]\right\}$	$f(a) = (8/27)\,A$
Brody	$f(t) = A\,(1 - B\,e^{-kt})$	doesn't exist	doesn't exist
Logistic	$f(t) = \frac{A}{1 + B\,e^{-kt}}$	$a = \frac{1}{k}\,log\left(\frac{A}{A_0} - 1\right)$	$f(a) = (A/2)$
Gompertz	$f(t) = A\,e^{-B\exp(-kt)}$	$a = \frac{1}{k}\,log\left\{\log\left(\frac{A}{A_0}\right)\right\}$	$f(a) = (A/e)$
Generalized Weibull	$f(t) = A\left[1 - B\,e^{-k\left(\frac{t-\mu}{\delta}\right)^v}\right]$	$a = \mu + \delta\left(\frac{v-1}{kv}\right)^{\frac{1}{v}}$	$f(a) = A\left[1 - Be^{-\left(1-\frac{1}{v}\right)}\right]$
Weibull	$f(t) = 1 - e^{-\left(\frac{t-\mu}{\delta}\right)^v}$	$a = \mu + \delta\left(1 - \frac{1}{v}\right)^{\frac{1}{v}}$	$f(a) = \left[1 - e^{-\left(1-\frac{1}{v}\right)}\right]$
Monomolecular	$f(t) = A\,(1 - B\,e^{-kt})$	doesn't exist	doesn't exist
Mitscherlich	$f(t) = A\,(1 - B\,e^{-kt})$	doesn't exist	doesn't exist

2.7. Gompertz Function

The Gompertz function [16] is defined as $f(t) = A\,e^{-B\exp(-kt)}$, $B = log\left(\frac{A}{A_0}\right)$.

Inflection Point

For Gompertz function, $f'(t)$ and $f''(t)$ are respectively given by $f'(t) = k\,f(t)\,\log\left(\frac{A}{f(t)}\right)$ and $f''(t) = k\ f'(t)\left[\log\left(\frac{A}{f(t)}\right) - 1\right]$. Here, we observe that $f''(t) = 0 \Leftrightarrow \left[\log\left(\frac{A}{f(t)}\right) - 1\right] = 0 \Leftrightarrow f(t) = \frac{1}{e}A$. This implies that

$$a = \left(\frac{1}{k}\right)\ \log\left\{\log\left(\frac{A}{A_0}\right)\right\} \tag{7}$$

Equation (7) is the point of inflection since $f''(a) = 0$ at that point and also $f''(t) < 0$ and $f''(t) > 0$ are satisfied on the increasing and decreasing sides of that point. Hence, for Gompertz, the single point of inflection occurs when the growth reaches $(1/e)$ of its final weight at time t given by (7).

2.8. Generalized Weibull Functions

We have generalized the Weibull function and named as Generalized Weibull function and can be given by $f(t) = A\left[1 - B\,e^{-K\left(\frac{t-\mu}{\delta}\right)^{\nu}}\right]$ where $B = 1 - \frac{A_\mu}{A}$, in the same notations used in this paper.

Inflection Point

For Generalized Weibull growth function, $f'(t)$ and $f''(t)$ are respectively given by $f'(t) = k\left(\frac{\upsilon}{\delta}\right)[A - f(t)]\left(\frac{t-\mu}{\delta}\right)^{\nu}$ and $f''(t) = \left(\frac{1}{\delta}\right)\left(\frac{t-\mu}{\delta}\right)^{-1}\left[(\nu - 1) - k\nu\left(\frac{t-\mu}{\delta}\right)^{\nu}\right] f'(t)$. We now observe $f''(t) = 0 \Leftrightarrow \left[(\upsilon - 1) - k\nu\left(\frac{t-\mu}{\delta}\right)^{\nu}\right] = 0 \Leftrightarrow f(t) = A\left[1 - Be^{-\left(1-\frac{1}{\nu}\right)}\right]$ which implies

$$a = \mu + \delta\ \left(\frac{\nu-1}{k\nu}\right)^{\frac{1}{\nu}} \tag{8}$$

Equation (8) is the point of inflection, since $f''(a) = 0$ at that point and also $f''(t) < 0$ and $f''(t) > 0$ are satisfied on the increasing and decreasing sides of the point a. Hence, for Generalized Weibull, the single point of inflection occurs when the growth reaches the weight $f(a) = A\ \left[1 - Be^{-\left(1-\frac{1}{\nu}\right)}\right]$ at time $t = a$ given by (8).

2.9. Weibull Functions

The Weibull growth model [27] is given as $f(t) = 1 - e^{-\left(\frac{t-\mu}{\delta}\right)^{\nu}}$. The inflection point is derived as follows:

$$f'(t) = k\ \left(\frac{\upsilon}{\delta}\right)\ [A - f(t)]\ \left(\frac{t-\mu}{\delta}\right)^{\nu}$$

$$f''(t) = \left(\frac{1}{\delta}\right)\ \left(\frac{t-\mu}{\delta}\right)^{-1}\left[(\nu - 1) - k\nu\ \left(\frac{t-\mu}{\delta}\right)^{\nu}\right]\ f'(t).$$

Here, we observe that $f''(t) = 0 \Leftrightarrow \left[(\upsilon - 1) - k\nu\ \left(\frac{t-\mu}{\delta}\right)^{\nu}\right] = 0 \Leftrightarrow f(t) = A\ \left[1 - Be^{-\left(1-\frac{1}{\nu}\right)}\right]$ and this implies

$$a = \mu + \delta\ \left(\frac{\nu-1}{k\nu}\right)^{\frac{1}{\nu}} \tag{9}$$

Equation (9) is the point of inflection since $f''(a) = 0$ at that point and also $f''(t) < 0$ and $f''(t) > 0$ are satisfied on the increasing and decreasing sides of that point. Hence, at the single point of inflection, growth reaches $f(a) = A\ \left[1 - Be^{-\left(1-\frac{1}{\nu}\right)}\right]$. This fact can also be verified by directly substituting $A = 1,\ B = 1,\ k = 1$ in the inflection point of Generalized Weibull.

2.10. Monomolecular and Mitscherlich Functions

Here we show that the monomolecular growth function takes same form as Brody. Monomolecular growth function is defined (France et al, 1996), in its original notations, as $w = w_f - (w_f - w_0)e^{-\lambda t} = w_f\left[1 - \left(1 - \frac{w_0}{w_f}\right)e^{-\lambda t}\right]$ where w is the growth function at time t, w_f is the final (mature) value, $w = w_0$ at $t = 0$ is the initial value and λ is rate of growth. This function can be expressed as Brody function with notations $w = f(t), w_f = A,\ w_0 = A_0\ , B = 1 - \frac{w_0}{w}, \lambda = k$ as $f(t) = A\ (1 - Be^{-kt})$.

Mitscherlich growth function [28] is defined, in its original notations, as $y = \alpha\left[1 - e^{-\beta(t+\varrho)}\right]$ where y is the growth function at time t, α is the final (mature) growth, ϱ is a constant and β is rate of growth. The Mitscherlich function can be expressed as Brody with notations $y = f(t), \alpha = A, \beta = k, B = e^{-k\varrho}$ as $(t) = A\ (1 - Be^{-kt}\ )$. Not that the integral constant becomes $log(\mathrm{AB}) = -\beta\delta + \log\alpha$.

Here we observe that all the three models viz., Brody, Monomolecular and Mitscherlich similar functions. With same argument as for Brody, both Monomolecular and Mitscherlich do not have any inflection points.

3. Conclusions

It is shown that inflection point exist for each of the functions: generalized logistic, particular case of generalized logistic, Richards, Von Bertalanffy, logistic, Gompertz, generalized Weibull and Weibull. Brody, Monomolecular and Mitscherlich curves do not have any inflection points. Formula for inflection point of each of the commonly known growth functions is derived as a function of model parameters. Given the respective parameters of the growth function, one can compute the inflection point and also the corresponding value of the growth function. The model parameters and the corresponding inflection point are of statistical interest.

References

[1] Paine, C.E.T., Marthews, T.R., Vogt, D.R., Purves D., Rees M., Hector, A., Turnbull, L.A., "How to fit nonlinear plant growth models and calculate growth rates: an update for ecologists", *Methods in Ecology and Evolution*, Vol. 3(2012), 245–256 doi: 10.1111/j.2041-210X.2011.00155.x

[2] Ayele T.,"Simulation study of the commonly used mathematical growth models", *J. Ethio. Stat. Assoc.*, Vol. 17(2008), 44-53.

[3] Edwards, Jr., C.H. and Penney, D.E., "Calculus with analytic geometry", Printice Hall International, New Jersey, 1994.

[4] Fischer, S., "The role of macroeconomic factors in growth", *Journal of Monetary Economics*, Vol. 32(1993): 485 – 512.

[5] Bawa, S. and Abdullahi, I.S., "Threshold effect of inflation on economic growth in Nigeria", *CBN Journal of Applied Statistics*, Vol. 3(2011), No.1, 43-63.

[6] Ahmad, N.; Khan, M.G.M.; Rafi, L.S., "Analysis of an inflection s-shaped software reliability model considering log-logistic testing - effort and imperfect debugging", *International Journal of Computer Science and Network Security*, Vol. 11(2011), No. 1, 161-171.

[7] Ahmad, N., Khan, M.G.M., Rofi, L.S., "A study of testing-effort dependent inflection s-shaped software reliability growth models with imperfect debugging", *International Journal of Quality and Reliability Management*, Vol. 27(2010), No. 1, 89-110.

[8] Ahmad, N.; Khan, M.G.M. Rafi, L.S., "Inflection s-shaped software reliability growth models with testing-effort functions", VI International Symposium on Optimization and Statistics (ISOS-2008), Aligarh Muslim University, Aligarh, India, December 29-31, 2008.

[9] http://epp.eurostat.ec.europa.eu/statistics_explained/index.php/Population_projections (accessed on August 2013).

[10] http://www.td.com/document/PDF/economics/special/RecentMortgageRateHikesInCanada.pdf (accessed on August 2013).

[11] Eberhardt, L.L. and Breiwick, J.M.,"Models for population growth curves", International Scholarly Research Network, ISRN Ecology (2012), doi:10.5402/2012/815016.

[12] Ersoy, İ.E., Mendeş, M., Keskin, S., "Estimation of parameters of linear and nonlinear growth curve models at early growth stage in California Turkeys". Arch. Geflügelk. Vol. 71(2007), No.4, 175–180.

[13] France, J., Dijkstra, J., MDhanoa, M.S., "Growth functions and their application in animal science", *Ann. Zootechn* Vol. 45(1996), 165-174.

[14] Brown, J.E., Fitzhugh, Jr., H.A., Cartwright, T.C., "A comparison of nonlinear models for describing weight-age relationship in cattle", *J. Animal Sci.* Vol. 42(1976), 810-818.

[15] Brody, S., "Bioenergetics and growth", Rheinhold Pub. Corp. N.Y, 1945.

[16] Winsor, C.P., "The Gompertz curve as a growth curve", Proc. National Academy of Science, Vol. 18(1932), No.1.

[17] Robertson, T.B., "On the normal rate of growth of an individual and its biochemical significance", Arch Entwicklungsmech Org Vol. 25(1906), 581-614.

[18] Lei, Y.C. and Zhang, S.Y., "Features and partial derivatives of Bertalanffy-Richards growth model in forestry", *Nonlinear Analysis: Modelling and Control*, Vol. 9(2004), No. 1, 65-73.

[19] Zeide, B., "Analysis of growth equations", *Forest Science*, Vol. 39(1993), No. 3, 594-616.

[20] Fekedulegn, D., Mac Siurtain, M.P., Colbert, J.J., "Parameter estimation of nonlinear growth models in forestry", *Silva Fennica* Vol. 33(1999), No. 4, 327-336.

[21] Koya, P.R., Goshu, A.T., "Solutions of rate-state equation describing biological growths", *American Journal of Mathematics and Statistics*, Vol. 3(2013), No. 6, 305-311, http://dx.doi.org/ 10.5923/j.ajms.20130306.02.

[22] Koya, P.R., Goshu, A.T., "Generalized mathematical model for biological growths", *Open Journal of Modelling and Simulation*, Vol. 1(2013), 42-53 http://dx.doi.org/10.4236/ojmsi. 2013.14008.

[23] http://en.wikipedia.org/w/index.php?oldid=472125857 (accessed on December 2012).

[24] Richards, F.J., "A flexible growth function for empirical use", *J. Exp. Bot.* Vol. 10(1959), 290-300.

[25] Bertalanffy, von L., "Quantitative laws in metabolism and growth", Quart. Rev. Biol. Vol. 3(1957), No. 2, 218.

[26] Nelder, J.A., "The fitting of a generalization of the logistic curve", *Biometrics* Vol. 17(1961), 89-110.

[27] Rawlings, J.O., Pantula, S.G., Dickey, D.A., "Applied regression analysis: a research tool". Second Edition, 1998.

[28] Mombiela, R.A. and Nelson, L.A., "Relationships among some biological and empirical fertilizer response models and use of the power family of transformations to identify an appropriate model", *Agronomy Journal*, Vol. 73(1981), 353-356.