American Journal of Modern Physics
2016; 5(2-1): 17-36
http://www.sciencepublishinggroup.com/j/ajmp
doi: 10.11648/j.ajmp.2016050201.12
ISSN: 2326-8867 (Print); ISSN: 2326-8891 (Online)

Studies on Santilli's Isonumber Theory

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To cite this article:
Arun S. Muktibodh. Studies on Santilli's Isonumber Theory. *American Journal of Modern Physics*. Special Issue: Issue II: Foundations of Hadronic Mechanics. Vol. 5, No. 2-1, 2016, pp. 17-36. doi: 10.11648/j.ajmp.2016050201.12

Received: August 10, 2015; **Accepted:** August 11, 2015; **Published:** May 18, 2016

Abstract: Beginning with studies in the 1980s at the Department of Mathematics of Harvard University, the Italian-American scientist R. M. Santilli discovered new realizations of the abstract axioms of numeric fields with characteristic zero, based on an axiom-preserving generalization of conventional associative product and consequential positive-definite generalization of the multiplicative unit, today known as Santilli *isonumbers* [1], and the resulting novel numeric fields are known as Santilli *isofields*. By remembering that 20th century mathematics was formulated on numeric fields, their generalization into isofields stimulated a corresponding generalization of all of 20th century mathematics and its application to mechanics, today known as Santilli *isomatheatics* and *isomechanics*, respectively, which is used for the representation of extended-deformable particles moving within physical media under Hamiltonian as well as contact non-Hamiltoian interactions. Additionally, Santilli discovered a second realization of the abstract axioms of a numeric field, this time with arbitrary (non-singular) negative definite generalized unit and related multiplication, today known as Santilli *isodual isonumber* [1] that have stimulated a second covering of 20th century mathematics and mechanics known as Santilli *isodual isomathematics* and *isodual isomechanics.* The latter methods are used for the classical as well as operator form of antimatter in full democracy with the study of matter. In this paper, we present a comprehensive study of Santilli's epoch making discoveries of isonumbers and their isoduals along with their application to isomechanics and its isodual for matter and antimatter, respectively.

Keywords: Isonumber, Isodual Number, Isodual-Isonumber, Genonumber

1. Introduction

As it is well known, modern mathematics has a strong foundation on number theory, algebraic structures such as groups, rings, algebra, vector spaces and related methods have found vast applications in all quantitative sciences. More general structures like groupoids, semigroups, monoids, quasigroups and loops were also being studied in 20th century, although their applications in quantitative sciences are under development. The detailed consolidated account of these generalized structures is found in Survey of Binary Systems by R.H.Bruck [2].

While the scientific discoveries and mathematical knowledge were moving hand in hand, towards the end of 20th century there were few mathematically unexplained physical phenomena in Quantum Physics and Quantum Chemistry. These new physical situations could not be faithfully described by the existing mathematical structures and called for more generalized mathematical structures.

It was Enrico Fermi, [3] beginning of chapter VI, p.111 said "..... there are some doubts as to whether the usual concepts of geometry hold for such small region of space." His inspiring doubts on the exact validity of quantum mechanics for the nuclear structure led to the genesis of the whole new kind of generalized mathematics, called isomathematics and generalized mechanics, called as Hadronic mechanics.

In fact, the prevailing Newtonian and Einsteinian 'Dynamical systems' called as 'Exterior Dynamical systems' which are characterized as 'local', 'linear' 'Lagrangian' and 'Hamiltonian' could not accommodate these obscure situations. *Thus it was the pressing demand of time to formulate new mathematical theory which could deal with the obscure phenomena and develop a new physical theory.* This stupendous task was taken up by the Italian-American theoretical physicist Ruggero Maria Santilli, President of Institute for Basic Research, Palm Harbor, Florida, USA and did the pioneering work by defining axiom-preserving, nonlinear, nonlocal and noncanonical isotopies of conventional mathematical structures, including units, fields,

vector spaces, transformation theory, algebras, groups, geometries, Hilbert spaces etc. while at Department of Mathematics of Harvard University in the early 80's. Prof. Santilli has rightly said;

"There can not be really new physical theories without

really new mathematics, and there can not be really

new mathematics without new numbers".

The founders of analytic mechanics, such as Lagrange, Hamilton [4] and others classified dynamical systems in to two kinds. First one is the 'Exterior Dynamical system' and the second one is the more complex but generalized 'Interior Dynamical system'.

However, over a period of time the the above distinction was abandoned preventing the identification of limitations of the prevailing mathematical and physical theories. One can easily notice that *Lie's Theory* is exactly applicable to the exterior dynamical systems. It was Prof. Santilli who at the Department of Mathematics of Harvard University, for the first time, drew the attention of the scientific community towards the crucial distinction between exterior and interior dynamical systems and presented insufficiencies of prevailing mathematical and physical theories by submitting the so-called *axiom-preserving, nonlinear, nonlocal, and noncanonical isotopies of Lie's theory* [5] under the name *Lie Isotopic theory*. Further generalization as *Lie-admissible theory* [6,7] was also achieved by him.

Exterior Dynamical Systems: In this system Point-like particles are moving in a homogeneous and isotropic vacuum with local-differential and potential-canonical equations of motion. These are linear, local, Newtonian Lagrangian and Hamiltonian. Conventional Mathematical structures such as Algebras, Geometries, Analytical Mechanics, Lie Theory can faithfully represent these systems.

Interior Dynamical Systems: In this system we consider extended non-spherical deformable particles moving within non-homogeneous anisotropic physical medium. These are non-linear, non-local, non-Newtonian, non-Lagrangian and non-Hamiltonian. The mathematical structures needed to describe these systems are most general possible which are axiom preserving; non-linear and non-local formulations of current mathematical structures.

During a talk at the conference *Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany, in 1980, Ruggero Maria Santilli submitted new numbers based on certain axiom preserving generalization of the multiplication, today known as *isotopic numbers* or *isonumbers*[1] in short. This generalization induced the so-called isotopies of the conventional multiplication with consequential generalization of the multiplicative unit, where the Greek word "isotopy" from the Greek word ``ισοσ τοποσ " implied the meaning "same topology" [8,9]. Subsequently, Ruggero Maria Santilli submitted a **new conjugation**, under the name *isoduality* which yields an additional class of numbers, today known as ***isodual isonumbers*** [1].

The discovery of isonumbers was made with the specific need of quantitative representation of the transition from Exterior Dynamical Systems to Interior Dynamical System.

It should be quite clear that there can not be new numbers without new fields. This led Santilli to define 'Isofield' which is the first new algebraic structure defined by him. This concept of 'Isofield' further led to a plethora of new isoalgebraic structures and a whole new 'Isomathematics' which is a step further in Modern Mathematics. Subsequently, 'Isomathematics' has grown in to a huge tree with various branches like 'Isofunctional Analysis', 'Isocalculus', 'Isoalgebra', isocryptography etc.

Prof. Santilli attracted great attention from academic community at Chinese Academy of Sciences during a workshop in China on August 23, 1997. Since then Prof Santilli and his associates in various countries around the world have produced numerous papers, monographs, conference proceedings which cover approximately 10,000 pages of research work.

Today Number theory has advanced as an important branch of axiomatized mathematics with highly sophisticated applications in the Modern world of computer science and information technology. After some advances in 19th century due to Gauss [10], Abel [11], Hamilton [4], Cayley [12], Galois [13] and others, major important advances were made during 20th century which included axiomatic formulation, the algebraic number theory [14].

The classification of all normed algebras with identity, over reals, in view of the previous studies by Hurwitz[15], Albert [16], and N.Jacobson [17] may be expressed in the following important Theorem.

Theorem 1.1. All possible normed algebras with multiplicative unit over the field of real numbers are given by algebras of dimension 1 (real numbers), 2 (complex numbers), 4 (quaternions), and 8 (octonians).

In this comprehensive presentation of the development of 'Isonumber theory' we cover the following important aspects of fundamental importance as formulated by Prof. R. M. Santilli [18], [1].

Starting with the brief background of the origin of 'isounit' and isofield, we present the theory of isonumbers, pseudoisonumbers, "hidden numbers" and their isoduals. Genonumbers, pseudogenonumbers and their isoduals are also of fundamental importance. We will study the isotopies and isodualities of the notions of numbers, fields and normed algebras with unit ref.[1]. In short, in this paper we are going to study the properties of isonumbers and their isoduals [1].

In his study Santiili has taken into account the four normed algebras over reals as given in the above theorem. The isotopic lifting of these algebras give rise to isotopies of normed algebras with multiplicative unit of dimension 1,2,4 and 8 which includes realization of 'isoreal numbers', 'isocomplex numbers', 'isoquaternions' and 'isooctonions'. Isodualities of these structures give isodual isonumbers.

The mathematical non-triviality of these structures is evident due to lack of unitary equivalence of isotopic and genotopic theories to conventional ones, non-applicability of

trigonometry and some other aspects. On the other hand, the physical non-triviality of these structures emerges from the fact that this theory of isonumbers is at the foundations of Li-isotopic theory used successfully to study nonlinear, nonlocal, and nonhamiltonian dynamical systems. The more general Lie-admissible theory emerges from the more general genonumbers.

In a nutshell, the theory of isonumbers is at the foundation of current studies of nonlinear-nonlocal-nonhamiltonian systems in nuclear, particle and statistical physics, superconductivity and other fields.

1.1. Origin of Isonumbers

The concept of 'Isotopy' plays a vital role in the development of this new age mathematics ref. R. H. Bruck [2] and [19].

The first and foremost algebraic structure defined by Santilli is 'isofield'. Elements of an isofield are called as 'isonumbers'. The conversion of unit 1 to the isounit $\hat{1}$ is of paramount importance for further development of 'Isomathematics'.

The reader should be aware that there are various definitions of "fields" in the mathematical literature [20], [21], [22] and [14] with stronger or weaker conditions depending on the given situation. Often "fields" are assumed to be associative under the multiplication.

i.e.

$$a\times(b\times c)=(a\times b)\times c \quad \forall a,b,c\in F .$$

We formally define an isofield [23], [24] as follows.

Definition 1.1 *Given a "field" F, here defined as a ring with with elements $a,b,c\ldots$, sum $a+b$, multiplication ab, which is commutative and associative under the operation of conventional addition $+$ and (generally nonassociative but) alternative under the operation of conventional multiplication $\times$ and respective units 0 and 1, "Santilli's isofields" are rings of elements $\hat{a}=a\hat{1}$ where a are elements of F and $\hat{1}=T^{-1}$ is a positive- definite $n\times n$ matrix generally outside F equipped with the same sum $\hat{a}+\hat{b}$ of F with related additive unit $\hat{0}=0$ and a new multiplication $\hat{a}*\hat{b}=\hat{a}T\hat{b}$, under which $\hat{1}=T^{-1}$ is the new left and right unit of F in which case $\hat{F}$ satisfies all axioms of the original field.*

T is called the isoelement. In the above definitions we have assumed "fields" to be alternative, i.e.

$$a\times(b\times b)=(a\times b)\times b,\quad (a\times a)\times b=a\times(a\times b)\quad \forall a,b\in F .$$

Thus, "isofields" as per above definition are not in general isoassociative, i.e. they generally violate the isoassociative law of the multiplication, i.e.

$$\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c})=(\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c}, \quad \forall\hat{a},\hat{b},\hat{c}\in\hat{F}$$

but rather satisfy isoalternative laws.

The specific need to generalize the definition of "number" to 'real numbers', complex numbers, 'quaternions' and 'octonians' suggested the above definition. The resulting new numbers are 'isoreal numbers', isocomplex numbers, 'isoquaternions' and 'isooctonians' respectively, where 'isooctonians' are alternative but not associative.

The 'isofields' $\hat{F}=\hat{F}(\hat{a},+,\hat{\times})$ are given by elements $\hat{a},\hat{b},\hat{c}\ldots$ characterized by one-to-one and invertible maps $a\to\hat{a}$ of the original element $a\in F$ equipped with two operations $(+,\hat{\times})$, the conventional addition $+$ of F and a new multiplication $\hat{\times}$ called "isomultiplication" with corresponding conventional additive unit 0 and a generalized multiplicative unit $\hat{1}$, called "multiplicative isounit" under which all the axioms of the original field F are preserved.

Santilli has shown that the transition from exterior dynamical system to interior dynamical system can be effectively represented via the isotopy of conventional multiplication of numbers a and b from its simple possible associative form $a\times b$ in to the isotopic multiplication, or isomultiplication for short, as introduced in [8].

The lifting of the product $ab=a\times b$ of conventional numbers in to the form

$$a\hat{\times}b:=a\times T\times b \tag{1}$$

denoted by $\hat{\times}=\times T\times$, where T is a fixed invertible quantity for all possible a,b called isotopic element.

This isomultiplication then lifts the conventional unit 1 defined by $1\times a=a\times 1=a$ to the multiplicative isounit $\hat{1}$ defined by

$$\hat{1}\hat{\times}a=a\hat{\times}\hat{1}=a,\text{where}\quad \hat{1}=T^{-1} \tag{2}$$

Under the condition that $\hat{1}$ preserves all the axioms of 1 the lifting $1\to\hat{1}$ is an isotopy, i.e. the conventional unit 1 and the iso unit $\hat{1}$ (as well as the conventional product $a\times b$ and its isotopic form $a\hat{\times}b$) have the same basic axioms and coincide at the abstract level by conception. The isounit $\hat{1}$ is so chosen that it follows the axioms of the unit 1 namely; boundedness, smoothness, nowhere degeneracy, hermiticity and positive-definiteness. This ensures that the lifting $1\to\hat{1}$ is an isotopy and conventional unit 1 and the isounit $\hat{1}$ coincide at the abstract level of conception.

Thus, the isonumbers are the generalization of the conventional numbers characterized by the isounit and the isoproduct as defined above.

The liftings $a\to\hat{a}$, and $\times\to\hat{\times}$ can be used jointly or individually.

It is important to note that unlike isotopy of multiplication $\times\to\hat{\times}$, the lifting of the addition $+\to\hat{+}$ implies general loss of left and right distributive laws. Hence the study of such a lifting is the question of independent mathematical investigation.

The first generalization was introduced by Prof. Santilli when he generalized the real, complex and quaternion numbers [23], [24] based on the lifting of the unit 1 into isounit $\hat{1}$ as defined above. Resulting numbers are called isorealnumbers, isocomplex numbers and isoquaternion numbers.

In fact, this lifting leads to a variety of algebraic structures which are often used in physics. The following flowchart is self explanatory.

Isonumbers $\rightarrow$ Isofields $\rightarrow$ Isospaces $\rightarrow$ Isotransformations $\rightarrow$ Isoalgebras $\rightarrow$ Isogroups $\rightarrow$ Isosymmetries $\rightarrow$ Isorepresentations $\rightarrow$ Isogeometries etc.

The isounit is generally assumed to be outside the original field with all the possible compatible conditions imposed on it. For rudiments of isomathematics reader can refer to [1, 6, 7, 25].

The lifting of unit I to isounit $\hat{I}$ may be represented as, $I \rightarrow \hat{I}(t,r,\dot{r},p,T,\psi,\psi^{\dagger},\partial\psi,\partial\psi^{\dagger},\ldots)$. where t is time, r is the position vector, p is the momentum vector, ψ is the wave function and $\psi^{\dagger}$ are the corresponding partial differentials. The positive definiteness of the isounit $\hat{I}$ is assured by, $\hat{I}(t,r,\dot{r},p,T,\psi,\psi^{\dagger},\partial\psi,\partial\psi^{\dagger},\ldots) = \frac{1}{T} > 0$ where T is called the isotopic element, a positive definite quantity. The isonumbers are generated as, $\hat{n} = n \times \hat{I}$, $n = 0,1,2,3,\ldots$.

Isofields are of two types, isofield of first kind; wherein the isounit does not belong to the original field, and isofield of second kind; wherein the isounit belongs to the original field. The elements of the isofield are called as isonumbers. This leads to number of new terms and parallel developments of conventional mathematics.

2. Isounits and Their Isoduals

As stated earlier, the isonumbers and their product can first be introduced as the generalization of conventional numbers by equations (1) and (2) as above.

Prof. Santilli further, introduced *isodual isonumbers* [26, 27, 28] by lifting the isounit into the form

$$\hat{1}^{d} \hat{\times}^{d} a = a \hat{\times}^{d} \hat{1}^{d} = a, \text{where} \quad \hat{1}^{d} := -\hat{1} \tag{3}$$

called the *isodual isounit* following lifting of iso multiplication defined in (1) into the *isodual multiplication* called *isoduality* as

$$a \hat{\times} b \rightarrow a \hat{\times}^{d} b := a \times T^{d} \times b = -a \times T \times b = -a \hat{\times} b \quad \text{where} T^{d} = -T \tag{4}$$

The *isodual isonumbers* were first conceived as characterized by isodual multiplication (4) with respect to the multiplicative *isodual isounit* $\hat{1}^{d} = -\hat{1}$.

The significance of isonumbers and isodual isonumbers lies in fulfilling the specific physical needs refs [18, 29, 30, 31] as given below;

- In the exterior dynamical system ordinary particles moving in the vacuum are characterized by conventional numbers.
- In the interior dynamical system ordinary particles moving in the physical medium are characterized by isonumbers.
- In the exterior dynamical system ordinary antiparticles moving in vacuum are characterized by isodual numbers.

In the interior dynamical system the antiparticles moving in the physical medium are characterized by isodual isonumbers.

Interpretation of customary characterization of antiparticles via negative-energy solutions of Dirac's equations behave in an un-physical way when interpreted with respect to the same numbers and unit 1 of particles, forcing various hypothetical assumptions and postulates, where as, reinterpretation of antiparticles with same negative energy solutions when interpreted as belonging to the field of isodual numbers behave in a fully physical way ref [1]. This treatment of antiparticles with isodual numbers also leads to intriguing geometrical implications which predict another universe, called as *isodual universe*, interconnected to our universe via isoduality and identified by the isodualities of Riemannian geometry and their isoduals refs.[31, 24, 32]. Thus, the isodual theory emerged from the identification of negative units in the antiparticle component of the conventional Dirac equation and the reconstruction of the theory with respect to this new negative unit. Hence isoduality provides a mere reinterpretation of Dirac's original notion of antiparticle leaving all numerical predictions electro-weak interactions essentially unchanged.

In view of the definition of an isofield [1], we can say that an isofield is an additive abelian group equipped with a new unit (called isounit) and isomultiplication defined appropriately so that the resulting structure becomes a field.

If the original field is alternative then the isofield also satisfies weaker isoalternative laws as follows. $\hat{a} \hat{\times} (\hat{b} \hat{\times} \hat{b}) = (\hat{a} \hat{\times} \hat{b}) \hat{\times} \hat{b}$ and $\hat{a} \hat{\times} (\hat{a} \hat{\times} \hat{b}) = (\hat{a} \hat{\times} \hat{a}) \hat{\times} \hat{b}$.

We mention two important propositions by Santilli.

Proposition 2.1. The necessary and sufficient condition for the lifting (where the multiplication is lifted but elements not the elements) $F(a,+,\times) \rightarrow \hat{F}(a,+,\hat{\times}), \hat{\times} = \times T \times, \hat{1} = T^{-1}$ to be an isotopy (that is for $\hat{F}$ to verify all axioms of the original field F) is that T is a non-null element of the original field F.

Proposition 2.2. The lifting (where both the multiplication and the elements are lifted)

$$F(a,+,\times) \rightarrow \hat{F}(\hat{a},+,\hat{\times}), \hat{a} = a \times \hat{1}, \hat{\times} = \times T \times, \hat{1} = T^{-1}$$

constitutes an isotopy even when the multiplicative isounit $\hat{1}$ is not an element of the original field.

The above proposition guarantees the physically fundamental capability of generating Plank's unit ν of quantum mechanics into an integro-differential operator $\hat{1}$ for quantitative treatment of nonlocal interactions [33].

As the first application of the isotopies of numbers Santilli

considers the set $S=<in>$, the set of all purely imaginary numbers. This set is not closed ($i^2=-1\notin S$). On the other hand, the same set S represented as $\hat{S}(\hat{n},+,\hat{\times})$ with $\hat{n}=in$ constitutes an isofield. i.e. it verifies all the axioms of a field including closure under isomultiplication because $T=i^{-1}$ and $\hat{n}\hat{\times}\hat{m}=in\hat{\times}im=inm\in\hat{S}$.

This illustrates an important fact that, even when a given set does not constitute a field, there may exist an isotopy under which it verifies the axioms of a field.

As stated earlier the lifting of $+$ to $\hat{+}$ does not necessarily produce an isotopy of a given field. This lifting does not preserve the distributivity in the resulting set as stated in the following proposition 2.3.

Proposition 2.3 The lifting $F(a,+,\times)\to\hat{F}(\hat{a},\hat{+},\hat{\times})$ where

$$\hat{a}=a\times 1,\quad \hat{+}=+\hat{K}+,\quad \hat{0}=-\hat{K}=-\hat{K}\times\hat{1},$$
$$\hat{\times}=\times T\times,\quad \hat{1}=T^{-1}$$

where K is an element of the original field F and T is an arbitrary invertible quantity, is not an isotopy for all nontrivial values of the quantity $K\neq 0$, because it preserves all the axioms of proposition 2.1 except the distributive law.

Based on the failure of distributivity Santilli defines "pseudoisofields" as follows.

Definition 2.1 Let $\hat{F}(\hat{a},+,\hat{\times})$ be an isofield as defined above. Then the "pseudoisofields" $\hat{F}(\hat{a},\hat{+},\hat{\times})$ are given by the images of $\hat{F}(\hat{a},+,\hat{\times})$ under all possible liftings of the addition $+\to\hat{+}=+K+$, with additive isounit $\hat{0}=-\hat{K}=-K\times\hat{1}, K\neq 0$ in which case the elements $\hat{a}$ are called the "pseudoisonumbers".

For the algebra of isonumbers and isodual numbers readers are advised to refer [1, 34].

Images of field, isofield and pseudoisofield under the change of sign of the isounit $\hat{1}\to\hat{1}^d=-1$ is called the *Isotopic conjugation* or *isoduality* ref. [28, 29, 30].

Definition 2.2 Let $F(a,+,\times)$ be a field as per definition 1.1. Then the isodual field $F^d(a^d,+,\times^d)$ is constituted by the elements called "isodual numbers"

$$a^d:=a\times 1^d=-a \tag{5}$$

defined with respect to the "isodual multiplication" and related "isodual unit"

$$\times^d:=\times 1^d\times=-\times,\quad 1^d=-1. \tag{6}$$

Definition 2.3 Let $F(\hat{a},+,\hat{\times})$ be an isofield as per definition 1.1. Then the isodual isofield $F^d(\hat{a}^d,+,\hat{\times}^d)$ is constituted by the elements called "isodual isonumbers"

$$\hat{a}^d:=a^c\times 1^d=-a^c\times 1 \tag{7}$$

where a^c is the conventional conjugation of F (e.g. complex conjugation) defined in terms of the " isodual isomultiplication"

$$\hat{\times}^d:=\times T^d\times=-\hat{\times},\quad T^d=-T. \tag{8}$$

Definition 2.4 Let $F(\hat{a},\hat{+},\hat{\times})$ be a pseudofield $\hat{F}(\hat{a},\hat{+},\hat{\times})$ as per definition 2.1. Then the "isodual pseudofield" $F(\hat{a}^d,\hat{+}^d,\hat{\times}^d)$ is given by the image of the original isofield under isodualities (6) and (7) plus the additional isoduality

$$\hat{0}\to\hat{0}^d=-0 \tag{9}$$

and its elements $\hat{a}^d$ are called "isodual pseudonumbers.

2.1. Classes of Isofields

Kadeisvilli [35] classified isounits into five primary classes according to their usefulness.

- **CLASS I**: *Isounits*:- These are the isounits when they are sufficiently smooth, bounded, nowhere singular, Hermitian and positive-definite. This class is of primary use in physics for characterization of ordinary particles moving in interior physical conditions. This class represents the isotopy of the conventional unit.
- **CLASS II**: *Isodual Isounits*:- They are same as isounits except that they are negative-definite. Isodual isounits are used in physics to characterize antiparticles via reinterpretation of the negative energy solutions of Dirac's equation [31, 36]. They represent *isodual isotopy* according to *isodual conjugation*.
- **CLASS III**: *Singular Isounits*:- These occur when isounits are considered as a divergent limit, $\hat{1}\Rightarrow\pm\infty$. These are used to represent gravitational collapse into a singularity and other limit conditions ref.[37, 23].
- **CLASS IV**: *Indefinite Isounits* :- This class represents isounits which are sufficiently smooth, bounded, nowhere singular, Hermitian and can smoothly interconnect positive definite with negative definite values. These are particularly used in mathematics.
- **CLASS V**: *General Isounits*, when they are solely Hermitian:- This is the most general class which includes preceding ones and permits a large variety of additional realizations including those in terms of discrete structures, discontinous functions, distributions etc.

Isofields can be classified according to the isounits as defined above. They are;

1. Isofields.
2. Isodual isofields.
3. Singular isofields.
4. Indefinite isofields.
5. General isofields.

The following four fundamental numbers are generated depending upon the isofield we consider;

1. (a) Ordinary numbers: real numbers $R(n,+,\times)$,

complex numbers $C(c,+,\times)$, quaternions $Q(q,+,\times)$ and octonians $O(o,+,\times)$ which are used in the characterization of particles in vacuum.

(b) Isonumbers: isoreal numbers $\hat{R}(\hat{n},+,\hat{\times})$, isocomplex numbers $\hat{C}(\hat{c},+,\hat{\times})$, isoquaternions $\hat{Q}(\hat{q},+,\hat{\times})$ and isooctonians $\hat{O}(\hat{o},+,\hat{\times})$ which are used for the characterization of particles within the physical media.

(c) Isodual numbers: isodual real numbers $R^d(n^d,+,\times^d)$, isodual complex numbers $C^d(c^d,+,\times^d)$, isodual quaternions $Q^d(q^d,+,\times^d)$ and isodual octonians $O^d(o^d,+,\times^d)$ which are used in the characterization of antiparticles in vacuum.

(d) Isodual isonumbers: isodual isoreal numbers $\hat{R}^d(\hat{n}^d,+,\hat{\times}^d)$, isodual isocomplex numbers $\hat{C}^d(\hat{c}^d,+,\hat{\times}^d)$, isodual isoquaternions $\hat{Q}^d(\hat{q}^d,+,\hat{\times}^d)$ and isodual isooctonians $\hat{O}^d(\hat{o}^d,+,\hat{\times}^d)$ which are used for the characterization of particles within the physical media.

2. Genofield is the generalization of isofield with the selection of an ordering of the multiplication to the left or to the right and applied for the more general Lie-admissible branch of hadronic mechanics.
3. Pseudofields, and
4. Pseudogenofields are the further generalization based on lifting of addition which relaxes at least one axiom of conventional fields, and which do have applications in other fields.
5. Hyper numbers can be constructed from hyperstructures ref.[35].

2.2. Isospaces

Let $S(x,g,R(n,+,\times))$ be a metric (or pseudo metric) n-dimensional space with local coordinates x and (Hermitean) metric $g=g^\dagger$ over the field of reals $R(n,+,\times)$. Then the *isospace* $\hat{S}(x,\hat{g},\hat{R}(n,+,\hat{\times}))$ first introduced in [38] is characterized by;

$$\begin{aligned}&\hat{S}(x,\hat{g},\hat{R}(n,+,\hat{\times})):\quad \hat{g}=T\times g,\\ &\hat{\times}=\times T,\quad \hat{1}=T^{-1}.\end{aligned}\tag{10}$$

Also the *isodual isospace* [28] is given by;

$$\hat{S}^d(x,\hat{g}^d,\hat{R}^d(n^d,+,\hat{\times}^d)):\quad \hat{g}^d=T^d\times g=-T\times g,$$

$$\hat{\times}^d=\times T^d\times=-\times T\times,\quad \hat{1}^d=-\hat{1}.\tag{11}$$

Note that isospaces $\hat{S}(x,\hat{g},\hat{R}(n,+,\hat{\times}))$ coincide with spaces $S(x,g,R(n,+,\times))$ at the abstract level of conception. Spaces have the most general known curvature and integral character owing to the arbitrariness in the isotopic element T. The isometries $\hat{g}=T\times g$ have the most general possible, nonlinear, nonlocal, noncanonical dependence in all variables,

$$g=g(x)\to\hat{g}=T(t,x,\dot{x},\ddot{x},\ldots)\times g(x)=\hat{g}(t,x,\dot{x},\ddot{x},\ldots).\tag{12}$$

The isospaces which are most important for physical and mathematical applications are *isoeuclidean spaces* $\hat{E}(x,\hat{\delta},\hat{R})$, *isominkowski spaces* $\hat{M}(x,\hat{\eta},\hat{R})$ and *isoriemanian spaces* $\hat{R}(x,\hat{g},\hat{R})$. These are the foundations of the representation of nonlinear, nonlocal, and noncanonical interior systems in nonrelativistic and gravitational interior problems [31, 23].

Also, *pseudoisospaces* can be defined as the images $\hat{S}(x,\hat{g},\hat{R}(n,\hat{+},\hat{\times}))$ of the original space characterized by further lifting $+\to\hat{+}=+K+,\quad 0\to\hat{0}=-K$. Subsequently, *isodual pseudoisospaces* are also defined.

2.4. Isoalgebras

The concept of isoalgebra was fundamental in the correct description of interior dynamical systems. As conventional numbers constitute normed algebras with unit, isoalgebras were defined to represent isonumbers ref. [21, 8, 39]. An isovector space $\hat{U}$ with elements $A,B,C\ldots$ and isomultiplication $\hat{e}$ over an isofield $\hat{F}(a,+,\hat{\times})$ with elements a,b,c and isomultiplication $a\hat{\times}b$ with multiplicative isounit $\hat{1}=T^{-1}$ is called (associative or nonassociative) *isoalgebra* when it satisfies right and left scaler and distributive laws;

$$(a\hat{\times}A)\hat{\odot}B=A\hat{\odot}(a\hat{\times}B)=a\hat{\times}(A\hat{\odot}B).\tag{15}$$

$$(A\hat{\times}a)\hat{\odot}B=A\hat{\odot}(B\hat{\times}a)=(A\hat{\odot})B\hat{\times}a\tag{16}$$

$$\begin{aligned}&A\hat{\odot}(B+C)=A\hat{\odot}B+A\hat{\odot}C,\\ &(B+C)\hat{\odot}A=B\hat{\odot}A+C\hat{\odot}A\end{aligned}\tag{17}$$

for all the elements $A,B,C\in\hat{U}$ and $a,b,c\in\hat{F}$.

Note that the isoalgebra $\hat{U}$ may contain the matrices where as the iso field $\hat{F}$ can contain ordinary numbers.

The isoalgebra $\hat{U}$ is an *isodivision algebra* if the equation $A\hat{\times}x=B$ always admits a solution in $\hat{U}$, for nonzero A.

Isonorm can be defined in the following manner;

Let $\hat{e}_k$ be an "isobasis" of $\hat{U}$ over the isofield $\hat{F}(a,+,\hat{\times})$. Then the generic element $A\in\hat{U}$ can be written as $A=\sum_{k=1\ldots,m}n_k\hat{\times}\hat{e}_k$, with $n_k\in\hat{F}$ and $\hat{e}^{\hat{2}}=\sum_k\hat{e}_k\hat{\odot}\hat{e}_k=1$. The *isonorm* of $\hat{U}$ in the isobasis considered, is then given by;

$$\hat{A}=(\sum_{k=1\ldots,m} n_k^{\hat{2}})^{\frac{1}{2}}\times\hat{1}=(\sum_{k=1\ldots,m} n_k\hat{\times}n_k)^{\frac{1}{2}}\times\hat{1}\in\hat{F} \quad (18)$$

The isoalgebra $\hat{U}$ is said to be isoassociative if;

$$A\hat{\odot}(B\hat{\odot}C)=(A\hat{\odot}B)\hat{\odot}C,\forall A,B,C\in\hat{U} \quad \text{(isoassociative law)} \quad (19)$$

and

$$A^{\hat{2}}\hat{\odot}B=A\hat{\odot}(A\hat{\odot}B),A\hat{\odot}B^{\hat{2}}=(A\hat{\odot}B)\hat{\odot}B \quad \text{(isoalternative laws)} \quad (20)$$

The isoalgebra $\hat{U}$ is said to be *Lie-isotopic* when the isoproduct $\hat{e}$ satisfies Lie-algebra axioms (anticommutativity and Jacobi laws) in the following form;

$$A\hat{\odot}B=ATB-BTA,\quad A,T,B,\textbf{etc.}=\textbf{assoc.} \quad (21)$$

It is said to be *lie-admissible* when the antisymmetric bracket product is;

$$[A;B]:=A\hat{\odot}B-B\hat{\odot}A \quad (22)$$

and is *Lie-isotopic* as in the realization;

$$A\hat{\odot}B=ARB-BSA. \quad (23)$$

We shall be mainly interested in the isoassociative isonormed algebras with isounit $\hat{1}$ which can be extended to isoalternative algebras in order to include isooctonians.

Extension of U and $\hat{U}$ under the pseudofield $\hat{F}(a,\hat{+},\hat{\times})$ implies loss of distributive laws and hence do not remain algebras in the real sense, however, we call them *pseudoisoalgebras* ref.[39].

2.5. Isoreal Numbers and Their Isoduals

2.5.1. Real Numbers

Real numbers constitute a one-dimensional normed associative and commutative algebra $U(1)$ ref.[1].

Real numbers are realized ref.[8] as a one-dimensional real Euclidean space $E_1(x,,R(n,+,\times))$ which represents a straight line with origin at 0, local coordinates x, metric $\delta=1$, additive unit 0 and multiplicative unit 1. Another characterization of real numbers is defined by the isomorphism of the reals $R(n,+,\times)$ into the commutative one-dimensional multiplicative group of dilations $G(1)$ defined by;

$$x'=n\times x,\quad n\in R(n,+,\times),\quad x,x'\in E_1(x,\delta,R). \quad (24)$$

The basis is given by

$$e=1 \quad (25)$$

with the norm defined by

$$|n|=(n\times n)^{\frac{1}{2}}>0 \quad (26)$$

and

$$|n\times n'|=|n|\times|n'|. \quad (27)$$

2.5.2. Isodual Real Numbers

Isodual Real numbers constitute a one-dimensional isodual associative and commutative normed algebra $U^d(1)$ which is anti-isomorphic to $U(1)$ ref.[1].

Isodual real numbers are the conventional numbers n defined with respect to the isodual unit $1^d=-1$. The isodual conjugation of real numbers is then written as

$$n=n\times 1\rightarrow n^d=n\times 1^d=-n. \quad (28)$$

Note that, such a sign inversion occurs when the isodual real numbers are projected in the field of conventional real numbers. As a result, all the numerical values change sign under isoduality.

The one-dimensional real isodual Euclidean space $E^d(x,\delta^d,R^d(n^d,+,\times^d))$ is a straight line, with conventional additive unit 0, and isodual multiplicative unit $1^d=-1$. The $R^d(n^d,+,\times^d)$ represents the Euclidean space $E^d(x,\delta^d,R^d(n^d,+,\times^d))$. Also, the isodual dilations are defined by

$$x'=n^d\times^d x=n\times x \quad (29)$$

This establishes an isomorphism between $R^d(n^d,+,\times^d)$ and the isodual group of dilations $G^d(1)$ (the conventional group reformulated according to the multiplicative unit 1^d). Santilli points out that $E_1(x,\delta,R)$ and $E_1^d(x,\delta^d,R^d)$ are antiisomorphic and the same property holds for $G(1)$ and $G^d(1)$. Also, the isodual dilations coincide with dilations as defined above. Santilli further says that *"this could be the a reason for the lack of detection of isodual numbers until then."* ref.s [38, 27, 28].

In the isodual case, the isodual basis is given by

$$e^d=1^d \quad (30)$$

with isodual norm

$$|n|=(n\times n)^{\frac{1}{2}}\times 1^d=|n|\times 1^d=-|n|<0 \quad (31)$$

satisfying the axioms

$$| n^d \times n'^d |^d = | n^d |^d \times^d | n'^d |^d . \quad (32)$$

2.5.3. Isoreal Numbers

Isoreal numbers constitute a one-dimensional, isonormed isoassociative and isocommutative isoalgebra $\hat{U}(1) \approx U(1)$ ref.[1].

Isoreal numbers are the numbers $\hat{n} = n \times \hat{1}$ of an isofield of Class I, with isomultiplication defined by $\hat{\times} = \times T \times$ and isounit $\hat{1} = T^{-1} > 0$, generally outside the original field $R(n,+,\times)$. These can be represented as the isoeuclidean spaces $\hat{E}_{1,1}(x, \hat{\delta}, \hat{R}(\hat{n},+,\hat{\times}))$ with $\hat{\delta} = T\delta$, over $\hat{R}(\hat{n},+,\hat{\times})$ the isotopes of conventional one-dimensional Euclidean spaces $E_1(x, \delta, R)$.

Some of the important remarks are as follows.

- The conventional Euclidean space $E_1(x, \delta, R)$ and its isotopic covering $\hat{E}_{1,1}(x, \hat{\delta}, \hat{R})$ are locally isomorphic due to the joint liftings $\delta \to \hat{\delta} = T \times \delta$ and $1 \to \hat{1} = T^{-1}$.
- $\hat{E}_{1,1}(x, \hat{\delta}, \hat{R})$ is not a Riemannian space because of the intrinsic dependence of the isometric $\hat{\delta}$ on the derivatives $\dot{x}, \ddot{x}, \ldots$ as well as the fact that the basic unit is not the conventional quantity 1.
- However, $\hat{E}_{1,1}(x, \hat{\delta}, \hat{R})$ is a simple, yet bona-fide isoriemannian space [24], because $\hat{\delta} = T \times \delta = \hat{\delta}(t, x, \dot{x}, \ddot{x}, \ldots)$, where the local dependence is generally nonlinear, nonlocal and noncanonical in all variables.

In fact, the *one-dimensional isospace* $\hat{E}_{1,1}(x, \hat{\delta}, \hat{R})$ represents a one-dimensional generalization of conventional straight line, called as *isoline*. This is because of its intrinsically nonlinear, nonlocal and noncanonical metric $\hat{\delta}(t, x, \dot{x}, \ddot{x}, \ldots)$ with multiplicative isounit $\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \ldots)$. Then $\hat{R}_1(\hat{n},\hat{+},\hat{\times})$ can be realized via *isodilations* on $\hat{E}_1(x, \hat{\delta}, \hat{R})$ as;

$$x' = \hat{n} \hat{\times} x = n \times x, \quad (33)$$

which is isodual dilation and represents one-dimensional isogroup of isodilations $\hat{G}(1)$ same as the group $G(1)$ realized with respect to isounit $\hat{1}$.

Again, the *isobasis* is given by

$$\hat{e} = \hat{1} \quad (34)$$

with *isonorm* defined as;

$$\| \hat{n} \| := (n \times n)^{\frac{1}{2}} \times \hat{1} = | n | \times \hat{1} \quad (35)$$

which is the conventional norm only rescaled to the new unit $\hat{1}$. We then also have

$$\| \hat{n} \hat{\times} \hat{n}' \| = \| \hat{n} \| \hat{\times} \| \hat{n}' \|. \quad (36)$$

2.5.4. Isodual Isoreal Numbers

The isodual isoreal numbers are the realization of the one-dimensional isodual, isonormed, isoassociative and isocommutative isoalgebra $\hat{U}^d(1) \approx U^d(1)$ ***ref.[1].***

These are the isodual numbers

$$\hat{n}^d = n \times \hat{1}^d, \quad \hat{1}^d = -\hat{1} \quad (37)$$

in the isodual isofield $\hat{R}^d_{11}(\hat{n}^d,+,\times^d)$. These correspond to $\hat{E}^d_{11}(x, \hat{\delta}^d, \hat{R}^d)$ the isoeuclidean space of Class II $\hat{E}^d_{11}(x, \hat{\delta}^d, \hat{R}^d)$ of dimension one with *isodual isodilations*

$$x' = \hat{n}^d \hat{\times}^d x \quad (38)$$

coinciding with dilations (24). This also characterizes an isomorphism isodual isoreal numbers with the one-dimensional isodual isogroup $\hat{G}^d(1)$. The underlying isomorphism

$$E^d_1(x, \delta^d, R^d(n^d,+,\times^d)) \approx E^d_{11,1}(x, \hat{\delta}^d, \hat{R}^d(\hat{n}^d,+,\hat{\times}^d))$$

implies the $\hat{G}^d(1) \approx G^d(1)$.

The *isodual isobasis* is defined by

$$\hat{e}^d = \hat{1}^d \quad (39)$$

The isodual isonorm

$$\| \hat{n}^d \|^d := (n \times n)^{\frac{1}{2}} \times \hat{1}^d = -\| \hat{n} \| \quad (40)$$

verifies the axioms

$$\| \hat{n}^d \hat{\times}^d \hat{n}' \|^d = \| \hat{n}^d \|^d \hat{\times}^d \| \hat{n}' \|^d \quad (41)$$

2.6. Isocomplex Numbers and Their Isoduals

2.6.1. Complex Numbers

Complex numbers constitute a two-dimensional, normed associative and commutative algebra $U(2)$ *ref.[1].*

Complex numbers $c = n_0 + n_1 i$ where n_0 and n_1 are real numbers and i is an imaginary unit, are represented in a Gauss plane which is a realization of two-dimensional Euclidean space $E_2(x, \delta, R(n,+,\times))$ satisfying

$$x^2 = x^t \delta_{ij} x_j = x_1^2 + x_2^2 \in R(n,+,\times) \tag{42}$$

whose group of isometries is one dimensional Lie Group $O(2)$, the invariance of the circle. Hence, complex numbers can be represented via fundamental representation of $O(2)$ as follows.

A one-to-one correspondence between complex numbers and points in the Gauss plane can be obtained by following *dilative rotations*

$$z' = (x_1 + x_2 i)' = c \circ z = (n_0 + n_1 i) \circ (x_1 + x_2 i) \tag{43}$$

and multiplication

$$c \circ z = (n_0, n_1) \circ (x_1, x_2) = (n_0 x_1 - n_1 x_2, n_0 x_2 + n_1 x_1) \tag{44}$$

which preserve all the properties of a field.

Representation of a complex number via matrices has the following form

$$c := n_0 \times I_0 + n_1 \times I_1 = \begin{pmatrix} n_0 & n_1 \times i \\ n_1 \times i & n_0 \end{pmatrix} \tag{45}$$

where

$$-\bar{\mathrm{c}}^{\mathrm{d}} = \mathrm{c}, \mathrm{i}^{\mathrm{d}} = \mathrm{i} \tag{46}$$

which are well known as the identity and fundamental representation of $O(2)$.

Norm can also be defined as

$$| c | = | n_0 + n_1 \times i | := (\mathrm{Det} c)^{\frac{1}{2}} = (n_0^2 + n_1^2)^{\frac{1}{2}} \tag{47}$$

Also, the identification of basis in terms of matrices is $e_1 = I_0$ and $e_2 = I_1$.

2.6.2. Isodual Complex Numbers

Isodual complex numbers constitute a two-dimensional isodual, normed, associative and commutative algebra $U^d(2)$ ***anti-isomorphic to*** $U(2)$ *ref.[1].*

Isodual complex numbers are given by

$$C^d = \{(c^d, +, \times^d) | \times^d = -\times, i^d = -i, c^d = \bar{c} \times i^d = -\bar{c}, \bar{c} \in \bar{C}\} \tag{48}$$

where $\bar{c}$ is the complex conjugation. Thus, given a complex number $c = n_0 + n_1 \times i$, its isodual is given by

$$c^d = -\bar{c} = n_o^d + n_1^d \times \bar{i} = -n_0 - n_1 \times \bar{i} = -n_0 + n_1 \times i \in C^d. \tag{49}$$

Considering the group of isometries, the one-dimensional isodual Lie group $O^d(2)$ i.e. the image of O_2 under the lifting $I = diag.(1,1) \to I^d = diag.(-1,-1)$ of the two-dimensional isodual Euclidean space $E_2{}^d(x, \delta^d, R^d(n^d, +, \times^d))$ with basic invariant

$$\begin{aligned} x^{2d} &= x^t \delta^d x = x_i \delta_{ij}^d x_j = x_1^{2d} + x_2^{2d} = \\ & x_1 \times^d x_1 + x_2 \times^d x_2 = -x_1^2 - x_2^2 \in R^d(n^d, +, \times^d) \end{aligned} \tag{50}$$

isodual complex numbers can be characterized by the isorepresentation of $O^d(2)$.

Now, the image of the conventional plane under isoduality is the isodual Gauss plane. Also, a one-to-one correspondence between the points $P = (x_1, x_2)$ and complex numbers can be defined by *isodual dilative rotations* as

$$z' = (x_1 + x_2 \times i)' = c^d \circ^d z = (-n_0 + n_1 \times i) \circ^d (x_1 + x_2 \times i) \tag{51}$$

following the multiplication rules

$$\begin{aligned} & c^d \circ^d z = (-n_0, n_1) \circ^d (x_1, x_2) = \\ & (-n_0 \times x_1 \times n_1 \times x_2, -n_0 \times x_2 + n_1 \times x_1) \end{aligned} \tag{52}$$

which preserve all the properties of a field.

Isodual transformations form an isodual group $G^d(2)$ antiisomorphic to $G(2)$. Even the one-to-one correspondence between complex numbers and Gauss plane continues under isoduality.

Matrix representation of isodual complex numbers can be defined as

$$c^d := n_0^d \times i_0^d + n_1^d \times i_1^d = \begin{pmatrix} -n_0 & n_1 \times i \\ n_1 \times i & -n_0 \end{pmatrix}, \tag{53}$$

$$i_0^d = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{54}$$

with the isodual unit and isodual representations of $O^d(2)$ respectively.

The *isodual norm* can be defined as $| c^d |^d = | -n_0 + n_1 \times i |^d :=$

$$[\mathbf{Det}_R (c^d \times T^d)]^{\frac{1}{2}} \times i_0^d = (c^{-d} \times^d c^d)^{\frac{1}{2}} \times i_0^d$$

which may be written as

$$| c^d |^d = (c \times \bar{c}) \times i_0^d = (n_0^2 + n_1^2) \times i_0^d \tag{55}$$

and verifies the axioms

$$| c^d \circ c'^d |^d = | c^d |^d \times^d | c'^d |^d \in R^d, \quad c^d, c'^d \in C^d. \tag{56}$$

The *isodual basis* in terms of matrices is given by

$$e_i^d = i_0^d, \quad e_2^d = i_1^d. \tag{57}$$

2.6.3. Isocomplex Numbers

Isocomplex numbers constitute a two-dimensional, isonormed, isoassociative and isocommutative isoalgebras over the isoreals $\hat{U}(2) \approx U(2)$ *ref.[1].*

In this case we consider the isofield of isocomplex numbers

$$\hat{C} = \{(\hat{c}, +, \hat{\times}) \mid \hat{\times} = \times T \times, \hat{1} = T^{-1}, \hat{c} = c \times \hat{1}, \\ c \in C(c, +, \times)\} \tag{58}$$

withe generic element $\hat{c} = \hat{n}_0 + \hat{n}_1 \times i$. Here we need the two-dimensional isoeuclidean space of class I, $\hat{E}_{1,2}(x, \hat{\delta}, \hat{R}(\hat{n}, +, \hat{\times}))$. The most important realization used in the physical literature has the diagonalized and positive-definite isotopic element and isounit

$$T = diag.(b_1^2, b_2^2), \hat{1} = diag.(b_1^{-1}, b_2^{-2}), b_k > 0, k = 1,2. \tag{59}$$

with basic isoseparation

$$\hat{e}_1 = \hat{I}_0, \quad \hat{e}_{k+1} \\ = \hat{i}_k, \quad k = 1,2,3. \tag{60}$$

The group of isometries of this space is the Lie group $\hat{O}(2) \approx O(2)$, the group constructed with respect to the multiplicative isounit $\hat{1} = diag.(b_1^{-2}, b_2^{-2})$ which provides the invariance of all possible ellipses with semiaxes $a = b_1^{-2}, b = b_2^{-2}$ as the infinitely possible deformation of the circle $x^2 = x_1^2 + x_2^2 \in R(n, +, \times)$. Thus, isocomplex numbers are characterizable via fundamental representation of $\hat{O}(2)$.

Isocomplex numbers $\hat{c} = (\hat{n}_0, \hat{n}_1)$ can also be characterized to be the set of points $P = (\hat{x}_1, \hat{x}_2)$ on the *isogauss plane* on $\hat{E}_{1,2}(x, \hat{\delta}, \hat{R}(\hat{n}, +, \hat{\times}))$.

In fact, a one-to one correspondence between isocomplex numbers $\hat{C}(\hat{c}, +, \hat{\times})$ and the points on the isogauss plane can be defined via following *isodilative isorotations*

$$z' = (x_1 + x_2 \times i)' = \hat{c} \hat{\circ} z \tag{61}$$

characterized by the isomultiplication defined as

$$\hat{c} \hat{\circ} z = (\hat{n}_0, \hat{n}_1) \hat{\circ} (x_1, x_2) = \\ = ([(n_0 \times x_0) \times \hat{1} - \hat{\Delta}^{\frac{1}{2}} \times (n_1 \times n_2) \times \hat{1}], [(n_0 \times x_2) \times \hat{1} + (n_1 \times x_1) \times \hat{1}]),$$

with

$$\Delta = DetT = b_1^2 \times b_2^2 \tag{62}$$

Isocomplex numbers also admit following two-by-two matrix representation.

$$\hat{c} = \hat{n}_0 \times \hat{i}_0 \hat{+} n_1 \hat{i}_1 = \\ \begin{pmatrix} n_0 \times b_1^{-2} & i \times n_1 \times b_1^2 \times \Delta^{-\frac{1}{2}} \\ i \times n_1 \times b_2^2 \times \Delta^{-\frac{1}{2}} & n_0 \times b_1^{-2} \end{pmatrix} \tag{63}$$

where

$$\hat{1} = \hat{I}_0 = \begin{pmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{pmatrix}, \hat{I}_1 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i \times b_1^2 \\ i \times b_2^2 & 0 \end{pmatrix} \tag{64}$$

and

$$\Delta = \mathbf{Det}T = b_1^2 b_2^2 \tag{65}$$

which characterize the isounit and the fundamental (adjoint)representation of $\hat{O}(2)$ respectively.

The set of matrices (63) is closed under addition and isomultiplication. Also, each element possesses the isoinverse

$$\hat{c}^{-\hat{1}} = \hat{c}^{-1} \times \hat{1} \tag{66}$$

where $\hat{c}^{-\hat{1}}$ is the ordinary inverse. As a result, $\hat{S}(\hat{c}, +, \hat{\times})$ is an isofield with the local isomorphism $\hat{S}(\hat{c}, +, \hat{\times}) \approx \hat{C}(\hat{c}, +, \hat{\times})$. We note that the one-to-one correspondence between complex numbers and Gauss plane is preserved under isotopy. It is important know that the realization of complex numbers as matrices is not unique.

The *isonorm* is defined as

$$\|\hat{c}\| = [Det_R(\hat{c} \times T)]^{\frac{1}{2}} \times \hat{I}_0 = (n_0^2 + \Delta n_1^2)^{\frac{1}{2}} \times \hat{I}_0 \tag{67}$$

which readily verifies the axiom

$$\|\hat{c} \hat{\circ} \hat{c}'\| = \|\hat{c}\| \hat{\times} \|\hat{c}'\| \in \hat{R}, \quad \hat{c}, \hat{c}' \in \hat{C}. \tag{68}$$

The *isobasis* is given by

$$\hat{e}_1 = \hat{I}_0 \quad \hat{e}_2 = \hat{1}. \tag{69}$$

2.6.4. Isodual Isocomplex Numbers

The isodual isocomplex numbers constitute a two-dimensional, isodual, isonormed, isoassociative and isocommutative isoalgebras over the isodual isoreals $\hat{U}^d(2) \approx U^d(2)$ ref.[1].

Now the isodual isocomplex numbers are defined as

$$\hat{C}^d = \{(\hat{c}^d, +, \times^d) \mid \hat{c} = -\bar{c}\hat{1}^d, \hat{\times}^d = \times T^d \times, T^d \\ = -T, \hat{1}^d = T^{-1d}, c \in C(c, +, \times)\} \tag{70}$$

with generic element $\hat{c}^d = \hat{n}^d + \hat{n}_1{}^d \times i^d = -\hat{n}_0 + \hat{n}_1 \times i.$ Here we need a two-dimensional isodual isoeuclidean space $E^d_{II,2}(x, \delta^d, \hat{R}^d(n^d, +, \hat{\times}^d))$ with the realization

$$T^d = diag.(-b_1^2, -b_2^2), \hat{1}^d = diag.(-b_1^{-2}, -b_2^{-2}), b_k > 0, k = 1, 2, \quad (71)$$

with basic isoseparation

$$x^{2d} = (x^t \delta^d x) \times \hat{1}^d = (x_i \delta^d_{ij} x_j) \times \hat{1}^d = (-x_1 b_1^2 x_1 - x_2 b_2^2 x_2) \times \hat{1}^d \in \hat{R}(\hat{n}^d, +, \times^d), \quad (72)$$

whose group of isosymmetries is the isodual isoorthogonal group $\hat{O}^d(2) \tilde{=} O^d(2)$.

The *isodual isogauss plane* is defined as the set of points $P = (\hat{x}_1, x_2)$ on $\hat{E}_{1,2}(x, \delta^d, \hat{R}^d(\hat{n}^d, +, \hat{\times}^d))$ which characterize the isocomplex numbers $\hat{c} = (-\hat{n}_0, \hat{n}_1)$.

The correspondence between the isodual isocomplex numbers $\hat{C}^d(\hat{c}^d, +, \times^d)$ and the isodual gauss plane can be made one-to-one by the isodual isodilative isorotations

$$z' = (x_1 + x_2 \times i)' = \hat{c}^d \circ^d z \quad (73)$$

having rule for multiplication as

$$\hat{c} \hat{\circ}^d z = (\hat{n}_0, \hat{n}_1) \hat{\circ}^d (x_1, x_2) = \quad (74)$$

$$= [(-n_0 \times x_0) \times \hat{1} + \Delta^{\frac{1}{2}} \times (n_1 \times x_2) \times \hat{1}], [(-n_0 \times x_2) \times \hat{1} + (n_1 \times x_1) \times \hat{1}].$$

Isodual isoguass planes characterizes isodual isofield. Also the isodual isotransformations forms an isodual isogroup $\hat{G}^d(2) \approx G^d(2)$.

Isodual isocomplex numbers also admit the following two-by-two matrix representation.

$$\hat{c}^d = \hat{n}_0{}^d \times^d \hat{I}_0{}^d + n_1^d \times^d \hat{I}^d = \begin{pmatrix} -n_0 \times b_1^{-2} & i \times n_1 \times b_1^2 \times \Delta^{-\frac{1}{2}} \\ i \times n_1 \times b_2^2 \times \Delta^{-\frac{1}{2}} & -n_0 \times b_2^{-2} \end{pmatrix} \quad (75)$$

where

$$\hat{1}^d = \hat{I}_0{}^d = \begin{pmatrix} -b_1^{-2} & 0 \\ 0 & -b_2^{-2} \end{pmatrix}, \quad \hat{I}^d = \begin{pmatrix} 0 & -i \times b_1^2 \times \Delta^{-\frac{1}{2}} \\ -i \times b_2^2 \times \Delta^{-\frac{1}{2}} & 0 \end{pmatrix}. \quad (76)$$

This satisfies isomultiplication rule (74) characterizing the isodual isounit and fundamental representation of $\hat{O}^d(2)$.

The set of matrices representing isodual complex numbers $\hat{S}^d(\hat{c}^d, +, \times^d)$, is closed under addition and isomultiplication. Each element possesses the isodual isoinverse

$$(\hat{c}^{-1})^d = (\hat{c}^d)^{-1} \times \hat{I}^d. \quad (77)$$

As a result we get a local isomorphism $\hat{S}^d(\hat{c}^d, +, \times^d) \approx \hat{C}^d(\hat{c}^d, +, \times^d)$.

Now, the *isodual isonorm* can be defined as

$$\|\hat{c}\|^d = [Det_R(\hat{c}^d \times T^d)]^{\frac{1}{2}} \times \hat{I}_0{}^d = (n_0^2 + \Delta n_1^2)^{\frac{1}{2}} \times \hat{I}^d, \quad (78)$$

which verifies

$$\left\| \hat{c}^d \hat{\circ}^d \hat{c}'^d \right\|^d = \left\| \hat{c}^d \right\|^d \hat{\times}^d \left\| \hat{c}'^d \right\|^d \in \hat{R}^d, \quad \hat{c}^d, \hat{c}'^d \in \hat{C}^d. \quad (79)$$

The isodual isobasis is given by

$$\hat{e}_1{}^d = \hat{I}_0{}^d \quad \hat{e}_2{}^d = \hat{1}^d. \quad (80)$$

2.7. Isoquaternions and Their Isoduals

2.7.1. Quaternions

Quaternions constitute a normed, associative, non-commutative algebra of dimension 4 over reals $U(4)$ *ref.[1].*

Quaternions $q \in Q(q, +, \times)$ admit a realization in the complex Hermitean plane $E_2(z, \delta, C)$ with separation

$$E_2(z, \delta, C): \quad z ? z \quad z^{-i} \delta_{ij} z^j = z^{-1} z^1 + z^{-2} z^2, \quad \delta ? \equiv \delta \quad (81)$$

with basic (unimodular) invariant $SU(2)$. Hence quaternions have a fundamental representation $SU(2)$ by *Pauli's matrices*.

Quaternions Q can be realized as the pairs of complex numbers, $q = (c_1, c_2)$, $q \in Q$ and $c_1, c_2 \in C$ with multiplication $\circ$. Hermitean dilative rotation on $E_2(z, \delta, C)$ which leaves $z \dagger z$ invariant is given by

$$z'^1 = c_1 \circ z^1 + c_2 \circ z^2, \quad z'^2 = -\bar{c}_2 \circ z^1 + \bar{c}_1 \circ z^2, \quad (82)$$

where the dilation is represented by $\bar{c}_1 \circ c_1 + \bar{c}_2 \circ c_2 \neq 1$. These transformations form a group $G(4)$. This group is associative but noncommutative resulting into a one-to-one correspondence with quaternions.

Quaternions can be represented via matrices over the field of complex numbers $C(c, +, \times)$ as

$$q=\begin{pmatrix} c_1 & c_2 \\ -\bar{c}_2 & \bar{c}_1 \end{pmatrix} \tag{83}$$

with

$$c_1=n_0+n_3\times i,\quad c_2=n_1+n_2\times i \tag{84}$$

The matrix q admits the representation

$$q=n_0\times I_0+n_1\times i_1+n_2\times i_2+n_3\times i_3 \tag{85}$$

where I_0,i_1,i_2,i_3 are the Pauli's matrices

$$\begin{aligned} I_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ i_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \end{aligned} \tag{86}$$

with fundamental relations

$$i_n\times i_m=-\varepsilon_{nmk}\times i_k,\quad n\neq m,\quad n,m=1,2,3. \tag{87}$$

where ε_{nmk} is the tensor of rank three. The norm of the quaternion can be defined as

$$|q|=(q\dagger q')^{\frac{1}{2}}=(\sum_{k=1,2,3} n_k^2)^{\frac{1}{2}}, \tag{88}$$

satisfying

$$|q\circ q'|=|q|\times|q'|\in R,\quad q,q'\in Q \tag{89}$$

The basis is defined by

$$e_1=I_0,\quad e_{k+1}=i_k,\quad k=1,2,3. \tag{90}$$

2.7.2. Isodual Quaternions

Isodual quaternions constitute an isodual four-dimensional, normed associative and noncommutative algebra over the isodual reals $U^d(4)$ which is anti-isomorphic to $U(4)$ ref.[1].

Isodual quaternions $q^d\in Q^d(q^d,+,\times^d)$ can be represented via the isodual Hermitean Euclidean space

$$\begin{aligned} &E_2^d(z^d,\delta^d,C^d(c^d,+,\times^d)):(z^{-id}\delta_{ij}^d z^{jd})\times I^d \\ &=(-z^{-1}z^1-z^{-2}z^2)\times I^d\in R^d. \end{aligned} \tag{91}$$

Isodual complex numbers can also be realized via pairs of isodual complex numbers as $q^d=(c^d,c2^d), q^d\in Q^d,\quad c_1^d,c_2^d\in C^d$.

Also, the *isodual Hermitean dilative rotation* on $E_2^d(z^d,\delta^d,C^d(c^d,+,\times^d))$ leaving invariant $z^d\dagger\delta^d z^d$ is given by

$$\begin{aligned} z'^{1d} &= c_1^d\circ^d z^{1d}-\bar{c}_2^{\,d}\circ^d z^{2d}, \\ z'^{2d} &= c_2^d\circ^d z^{1d}+\bar{c}_1^{\,d}\circ^d z^{2d} \end{aligned} \tag{92}$$

where the dilation is represented by the value $\bar{c}_1^{\,d}\circ^d c_1^d+\bar{c}_2^{\,d}\circ^d c_2^d\neq -1$.

These transformations form an associative but noncommutative isodual group $G^d(4)$ which is in one-to-one correspondence with isodual quaternions $Q^d(q^d,+,\times^d)$.

As a result there is a matrix representation of isodual complex numbers over the field of isodual complex numbers $C^d(c^d,+,\times^d)$ as

$$q^d=\begin{pmatrix} c_1^d & -\bar{c}_2^{\,d} \\ c_2^d & \bar{c}_1^{\,d} \end{pmatrix} \tag{93}$$

under the condition

$$c_1^d=-n_0+n_3\times i,\quad c_2^d=-n_1+n_2\times i \tag{94}$$

where $-\bar{c}^d=c,\quad i^d=i$.

We can represent q^d as

$$\begin{aligned} q^d &= n_0^d\times^d I_0^d+n_1^d\times^d i_1^d+n_2^d\times^d i_2^d+n_3^d\times^d i_3^d= \\ &=-n_0\times I_0+n_1\times i_1+n_2\times i_2+n_3\times i_3 \end{aligned} \tag{95}$$

where i's are the Pauli's matrices. Note that Pauli's matrices change sign under isoduality although their product with isodual numbers is isoselfdual.

Isodual norm is then defined as

$$|q^d|=[Det_C(q^d\times T^d)]\times I^d=(-\sum_{k=0,1,2,3} n_k^2)^{\frac{1}{2}}\times I^d \tag{96}$$

satisfying

$$\begin{aligned} &|q^d\circ^d q'^d|^d=|q^d|^d\times^d|q'^d|^d\in R^d, \\ &q^d,q'^d\in Q^d. \end{aligned} \tag{97}$$

The *isodual basis* is defined as

$$e_1^d=I_0^d, e_{k+1}^d=i_k,\quad k=1,2,3. \tag{98}$$

2.7.3. Isoquaternions

Isoquaternions constitute a four-dimensional, isonormed, isoassociative, non-isocommutative isoalgebra over the

isoreals $\hat{U}(4) \approx U(4)$, *ref.[1].*

Isoquatrnions $\hat{q} \in \hat{Q}(\hat{q},+,\hat{\times})$ can be represented using two-dimensional, complex Hermitean isoeuclidean space of class I, $\hat{E}_{I,2}(\hat{z},\hat{\delta},\hat{C}), \hat{z}^k = z^k, \hat{z}_k = \hat{\delta}_{ki}\hat{z}^i, \hat{\delta} = \hat{T}\hat{\delta} = \hat{\delta}_{ij}$ on the isofield $\hat{C}(\hat{c},+,\hat{\times})$ with real separation given by

$$\hat{z}\hat{\dagger}\,\hat{\delta}\hat{z} = \bar{z}^1 b_1^2 z^1 + \bar{z}^2 b_2^2 z^2, \quad \hat{\delta}\hat{\dagger} \equiv \delta > 0, \tag{99}$$

with basic isotopic element and isounit

$$T = Diag.(b_1^2, b_2^2), \quad \hat{1} = Diag.(b_1^{-2}, b_2^{-2}), b_k > 0, \tag{100}$$

The (unimodular) invariance group of this space is the Lie-isotopic group $S\hat{U}(2)$. Isoquaternions can also be characterized by fundamental representation of $S\hat{U}(2)$ algebra. A Hermitean isodilative isorotation on $\hat{E}_{I,2}(\hat{z},\hat{\delta},\hat{C}(\hat{c},+,\hat{\times}))$ is given by

$$\hat{z}'^1 = \hat{c}_1 \hat{\circ} \hat{z}^1 + \hat{c}_2 \hat{\circ} \hat{z}^2, \quad \hat{z}'^2 = -\hat{c}_2 \hat{\circ} \hat{z}^1 + \hat{c}_1 \hat{\circ} \hat{z}^2, \tag{101}$$

where the dilation is represented by the value $\hat{c}_1 \hat{\circ} \hat{c}_1 + \hat{c}_2 \hat{\circ} \hat{c}_2 \neq \hat{1}$. Representation of isoquaternions into two-by-two matrices on $\hat{C}(\hat{c},+,\hat{\times})$ is characterized by the isorepresentations of the Lie-isotopic algebra $S\hat{U}(2)$ ref. [40, 41, 42]. These can be expressed in terms of the basic isounit

$$\hat{I} = \hat{1}_0 = \begin{pmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{pmatrix} \tag{102}$$

and fundamental representation of $S\hat{U}(2)$ as

$$\begin{aligned} &\hat{i}_1 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & ib_1^2 \\ ib_2^2 & 0 \end{pmatrix}, \quad \hat{i}_2 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & b_1^2 \\ -b_2^2 & 0 \end{pmatrix}, \\ &\hat{i}_3 = \Delta^{-\frac{1}{2}} \begin{pmatrix} ib_2^2 & 0 \\ 0 & -ib_1^2 \end{pmatrix} \end{aligned} \tag{103}$$

Note that the matrices above satisfy the properties of isotopic image

$$\begin{aligned} &\hat{i}_n \hat{\circ} \hat{i}_m = \Delta^{-\frac{1}{2}} \varepsilon_{nmk} \hat{1}_k, \quad n \neq m, \\ &n,m = 1,2,3, \quad \Delta = b_1^2 b_2^2, \end{aligned} \tag{104}$$

and hence are closed under commutators, which is a necessary condition for the existence of an isotopy. This results into a Lie-isotopic $S\hat{U}(2)$ algebra

$$[\hat{i}_n, \hat{\hat{i}}_m] := \hat{i}_n \hat{\circ} \hat{i}_m - \hat{i}_m \hat{\circ} \hat{i}_n = -2\Delta^{-\frac{1}{2}} \varepsilon_{nmk} \hat{i}_k. \tag{105}$$

Isoquaternions can be represented in the form

$$\hat{q} = n_0 I_0 + n_1 \hat{i}_1 + n_2 \hat{i}_2 + n_3 \hat{i}_3 =$$

$$\begin{pmatrix} (n_0 b_1^{-2} + \Delta^{-\frac{1}{2}} i n_3 b_2^2) & \Delta^{-\frac{1}{2}}(-n_2 + i n_1) b_1^2 \\ \Delta^{-\frac{1}{2}}(n_2 + i n_1) b_2^2 & (n_0 b_2^{-2} - \Delta^{-\frac{1}{2}} i n_3 b_1^2) \end{pmatrix}. \tag{106}$$

Note that the set $\hat{S}(\hat{q},+,\hat{\times}$ is a four dimensional vector space over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ which is closed under the operation of conventional addition and isomultiplication and hence, is an isofield. Thus, $\hat{S}(\hat{q},+,\hat{\times}) \approx \hat{Q}(\hat{q},+,\hat{\times})$.

The *isonorm* of the isoquaternions is defined as follows

$$\|\hat{q}\| = [Det_R(\hat{q}T)]^{\frac{1}{2}} \hat{I}_0, \tag{107}$$

and may be written as

$$\|\hat{q}\| = [n_0^2 + \Delta(n_1^2 + n_2^2 + n_3^2)]\hat{I}_0, \tag{108}$$

and then

$$\|\hat{q} \hat{\circ} \hat{q}'\| = \|\hat{q}\| \hat{\times} \|\hat{q}'\| \in \hat{R}, \quad \hat{q}, \hat{q}', \hat{\circ} \in \hat{Q} \tag{109}$$

The *isobasis* is defined as

$$\hat{e}_1 = \hat{I}_0, \quad \hat{e}_{k+1} = \hat{i}_k, \quad k = 1,2,3. \tag{110}$$

2.7.4. Isodual Isoquaternions

The isodual isoquaternions constitute a four-dimensional, isodual, isonormed, isoassociative, non-isocommutative isoalgebra over the isodual isoreals $\hat{U}(4) \approx U^d(4)$ *ref. [1].*

The *isodual isoquaternions* $\hat{q}^d \in \hat{Q}^d(\hat{q}^d,+\hat{\circ}^d)$ by a two-dimensional isodual complex Hermitean isoeuclidean space of class II over the isodual isocomplex field as

$$\begin{aligned} &E^d_{II,2}(\hat{z}^d, \delta^d, \hat{C}^d(c^d,+,\hat{\times}^d)) : \hat{z}^d \,?\, \hat{\delta}^d z^d \\ &z^{-1d} \hat{\times}^d z^{1d} + z^{-2d} \hat{\times}^d z^{2d} = -z^{-1} b_1^2 z^1 - z^{-2} b_2^2 z^2. \end{aligned} \tag{111}$$

having basic isodual isotopic element and *isodual isounit*

$$T^d = Diag.(-b_1^2, -b_2^2), \hat{1}^d = Diag.(-b_1^{-2}, -b_2^{-2}) \tag{112}$$

having invariance as the isodual Lie-isotopic group $S\hat{U}^d$. An *isodual Hermitean isodilative isorotation* on $E^d_{II,2}(\hat{z}^d, \delta^d, \hat{C}^d(c^d,+,\hat{\times}^d))$ is given by

$$\begin{aligned} \hat{z}'^{1d} &= \hat{c}_1^d \hat{\circ}^d \hat{z}^{1d} - \overline{c}_2^d \hat{\circ}^d z^{2d}, \\ \hat{z}'^{2d} &= \hat{c}_2^d \hat{\circ}^d \hat{z}^{1d} + \overline{c}_1^d \hat{\circ}^d \hat{z}^{2d}, \end{aligned} \tag{113}$$

where dilation is represented by

$\hat{c}_1^{\,d} \hat{\circ}^d \hat{c}_1^{\,d} + \bar{c}_2^{\,d} \hat{\circ}^d \hat{c}_2^{\,d} \neq \hat{1}^d$.

Isodual Isoquaternions can also be realized as the isodual isorepresentation of $S\hat{U}^d(2)$ and can be written as

$$\hat{q}^d = \hat{n}_0^{\,d} + \hat{n}_1^{\,d} \hat{\times}^d \hat{i}_1^{\,d} + \hat{n}_2^{\,d} \hat{\times}^d \hat{i}_2^{\,d} + \hat{n}_3^{\,d} \hat{\times}^d \hat{i}_3^{\,d} =$$

$$-\hat{n}_0 + \hat{n}_1\hat{i}_1 + \hat{n}_2\hat{i}_2 + \hat{n}_3\hat{i}_3 =$$

$$= \begin{pmatrix} (-n_0 b_1^{-2} + \Delta^{-\frac{1}{2}} i n_3 b_2^2) & \Delta^{-\frac{1}{2}}(-n_2 + i n_1) b_1^2 \\ \Delta^{-\frac{1}{2}}(n_2 + i n_1) b_2^2 & (-n_0 b_2^{-2} - \Delta^{-\frac{1}{2}} i n_3 b_1^2) \end{pmatrix} \quad (114)$$

Note that the set of all the matrices ${}^d(\hat{q}^d, +, \times^d)$ is an isofield and hence ${}^d(\hat{q}^d, +, \times^d) \approx \hat{Q}^d(\hat{q}^d, +, \hat{\times}^d)$.

The *isodual isonorm* is defined as

$$\hat{\|}\hat{q}^d\hat{\|}^d = [Det_R(\hat{q}^d T^d)]^{\frac{1}{2}} \hat{I}_0^{\,d} =$$

$$= [-n_0^2 - \Delta(n_1^2 + n_2^2 + n_3^2)]\hat{I}_0^{\,d}, \quad (115)$$

$$\left\|\hat{q}^d \hat{\circ}^d \hat{q}'^d\right\|^d = \left\|\hat{q}^d\right\|^d \hat{\times}^d \left\|\hat{q}'^d\right\|^d \in \hat{R}^d,$$
$$\hat{q}^d, \hat{q}'^d, \hat{\circ}^d \in \hat{Q}^d \quad (116)$$

The *isodual isobasis* is defined as

$$\hat{e}_1^{\,d} = \hat{I}_0^d \quad \hat{e}_{k+1}^d = \hat{i}_k^d, \quad k = 1,2,3. \quad (117)$$

2.8. Isooctonians and Their Isoduals

2.8.1. Octonians

Octonians constitute and eight-dimensional normed, non-associative and non-commutative , alternative algebra $U(8)$ ***over the field of reals*** $R(n,+,\times)$***.ref.[20, 21].***

Octonians $o \in O(o,+,\times)$ can be realized as two-dimensional quaternions $o = (q_1, q_2)$ with multiplication rules

$$o \circ o' = (q_1, q_2) \circ (q_1', q_2') =$$
$$(q_1 \circ q_1' + q_1 \circ q_2', -\bar{q}_1 \circ q_2' + \bar{q}_1 \circ q_2). \quad (118)$$

The *antiautomorphic conjugation* of an octonian is defined as

$$\bar{o} = (\bar{q}_1, -q_2). \quad (119)$$

The *norm* of an octonian is defined as

$$|o| := (\bar{o} \circ o)^{\frac{1}{2}} = |q_1| + |q_2|, \quad (120)$$

with the basic axioms

$$|o \circ o'| = |o| \times |o'| \in R, \quad o, o' \in O. \quad (121)$$

It is important to note that Octonions do not constitute a realization of the abstract axioms of a numeric field and, therefore, they do not constitute numbers as conventionally known in mathematics due to the non-associative character of their multiplication (see ref. [1]).

2.8.2. Isodual Octonians

The isodual octonians constitute an eight-dimensional isodual, normed, non-associative, and non-commutative algebra $U^d(8)$ ***over the isodual real numbers*** $R^d(n^d, +, \times^d)$ *ref. [1].*

Isodual octonians are defined as

$$o^d = (q_1^d, q_2^d) \quad (122)$$

over the isodual reals $R^d(n^d, +, \times^d)$. The isodual multiplication of isodual octonians is defined by

$$o^d \circ^d o'^d = (q_1^d, q_2^d) \circ^d (q_1'^d, q_2'^d) =$$
$$(q_1^d \circ^d q_1'^d - \bar{q}_1^d \circ^d q_2'^d, q_1^d \circ^d q_2'^d + \bar{q}_1^d \circ^d q_2^d). \quad (123)$$

The isodual antiautomorphic conjugation of an octonian is defined as

$$\bar{o}^d = (\bar{q}_1^d, -q_2^d). \quad (124)$$

The *isodual norm* of an octonian is defined as

$$|o| := (\bar{o} \circ o)^{\frac{1}{2}} = |q_1| + |q_2|, \quad (125)$$

with the basic axioms

$$|o^d \circ^d o'^d| = |o^d|^d \times^d |o'^d|^d \in R^d, \quad o^d, o'^d \in O. \quad (126)$$

2.8.4. Isodual Isooctonians

Isodual isooctonians form an eight-dimensional isodual, isonormed, non-isoassociative, non-isocommutative, but isoalternative isoalgebra $\hat{U}^d(8) \approx U^d(8)$ ***over the isodual isofield*** $\hat{R}^d(\hat{n}^d, +, \times^d)$***, ref. [43].***

Isodual isooctonians $\hat{o}^d \in \hat{O}^d(\hat{o}^d, +, \hat{\times}^d)$ can be defined as the pair of isoquaternions $\hat{o}^d = (\hat{q}_1^{\,d}, \hat{q}_2^{\,d})$ over the isodual isoreals $\hat{R}^d(\hat{n}^d, \hat{\times}^d)$ with the multiplication rule

$$\hat{o}^d \hat{\circ}^d \hat{o}'^d = (\hat{q}_1^{\,d}, \hat{q}_2^{\,d}) \hat{\circ}^d (\hat{q}_{1'}^{\,d}, \hat{q}_{2'}^{\,d}) =$$
$$(\hat{q}_1^{\,d} \hat{\circ} \hat{q}_1'^d - \tilde{q}_1^{\,d} \hat{\circ}^d \hat{q}_{2'}^{\,d}, q_1^d \hat{\circ}^d \hat{q}_{2'}^{\,d} + \hat{q}_1^{\,d} \hat{\circ}^d q_2^{\,d}) \quad (131)$$

The isodual isoantiautomorphism is defined as

$$\tilde{o}^d = (\tilde{q}^d, -\hat{q}_2^d) \quad (132)$$

The *isodual isonorm* is defined as

$$\left\|\hat{o}^d\right\|^d := (\tilde{o}^d \circ^d \hat{o}^d)^{\frac{1}{2}} \times \hat{1}^d = \left\|\hat{q}_1^d\right\|^d + \left\|\hat{q}_2^d\right\|^d \quad (133)$$

which readily verifies

$$\left\|\hat{o}^d \hat{\circ}^d \hat{o}'^d\right\| = \left\|\hat{o}^d\right\|^d \hat{\times}^d \left\|\hat{o}'^d\right\|^d \in \hat{R}^d, \quad \hat{o}^d, \hat{o}'^d \in \hat{O}^d. \quad (134)$$

Again it is important to note that Isodual isooctonians do not constitute a realization of the abstract axioms of a numeric field and, therefore, they do not constitute numbers as conventionally known in mathematics due to the non-associative character of their multiplication (see Ref. [1]).

3. Grand Unification of Numeric Fields

Isotopic generalization has brought about a grand unification of the conventional numbers into one single, abstract notion of isonumber. It is important to note that the unification of all numbers was conjectured by Prof. Santilli in numerous publications through out his research for many years. Finally it was proved by Kadeisville, Kamiya and Santilli ref.[40]. The following theorem is the main result in this regard.

Theorem 3.1. *Let* $\mathrm{F}(\mathrm{a},+,\times)$ *be the fields of real numbers, complex numbers and quaternions, respectively,* $\mathrm{F}^{\mathrm{d}}(\mathrm{a}^{\mathrm{d}},+,\times^{\mathrm{d}})$ *the isodual fields,* $\mathrm{a}^{\mathrm{d}} := \mathrm{a} \times 1^{\mathrm{d}} = -\mathrm{a}$ *the isofields, and* $\times^{\mathrm{d}} := \times 1^{\mathrm{d}} \times = -\times, \quad 1^{\mathrm{d}} = -1.$ *the isodual isofields as defined in the preceding section. Then all these fields can be constructed with the same methods for the construction of* $\mathrm{F}(\hat{\mathrm{a}},+,\hat{\times})$ *from* $\mathrm{F}^{\mathrm{d}}(\hat{\mathrm{a}}^{\mathrm{d}},+,\hat{\times}^{\mathrm{d}})$*, under the relaxation of the condition of positive-definiteness of the isounit, thus achieving a unification of all the fields, isofields and their isoduals into the single, abstract isofield of Class III, denoted by* $\boldsymbol{R}$.

3.1. Hidden Numbers of Dimension 3, 5, 6, 7

Based on the historical problem 'The four and eight square problem and division algebras' ref.[21], Prof. Santilli conjectured the possibility of 'Hidden numbers' of dimension 3, 5, 6 and 7'. The numbers studied by Santilli, namely, reals, complex, quaternions and octonians are the solution of the following problem.

$$(a_1^2 + a_2^2 + \ldots + a_n^2) \times (b_1^2 + b_2^2 + \ldots + b_n^2) = A_1^2 + A_2^2 + \ldots + A_n^2$$

with

$$A_k = \sum_{r,s} c_{krs} \times a_r \times b_s \quad (135)$$

where all the a's, b's and c's are elements of a field $F(a,+,\times)$ with conventional operations $+$ and $\times$. It is well known that the only possible solutions of the problem are of dimension 1, 2, 4 and 8. These facts are in corporated in the theorem 1.1, restated here

Theorem 3.2 All possible normed algebras with multiplicative unit over the field of real numbers are given by algebras of dimension 1 (real numbers), 2 (complex numbers), 4 (quaternions), and 8 (octonians).

The question posed by Santilli: Is 'Does the classification according to above theorem persist under isotopies, pseudoisotopies and their isodualities ?' or 'Is it incomplete ?' First, we investigate this problem for isotopies of the multiplication. The above problem, equation (135) is reformulated under the isotopies of the multiplication as follows.

The isotopic lifting of the multiplication

$$\times \to \hat{\times} = \times T \times, 1 \to \hat{1} = T^{-1} \quad (136)$$

transforms the problem (135) in to

$$(a_1^{\hat{2}} + a_2^{\hat{2}} + \ldots + a_n^{\hat{2}}) \hat{\times} (b_1^{\hat{2}} + b_2^{\hat{2}} + \ldots + b_n^{\hat{2}}) = A_1^{\hat{2}} + A_2^{\hat{2}} + \ldots + A_n^{\hat{2}} \quad (137)$$

with

$$A_k = \sum_{r,s} c_{krs} \hat{\times} a_r \hat{\times} b_s \quad (138)$$

where all the a's, b's and c's are elements of an isofield $\hat{F}(a,+,\hat{\times})$ in which $\hat{1}$ is an element of the original field, can be simplified to the conventional operations as

$$(a_1^2 + a_2^2 + \ldots + a_n^2) \times (b_1^2 + b_2^2 + \ldots + b_n^2) = T^{-2} \times (A_1^2 + A_2^2 + \ldots + A_n^2) \quad (139)$$

with

$$A_k = T^2 \times \sum_{r,s} c_{krs} \times a_r \times b_s. \quad (140)$$

Comparing the original problem and its isotopic conversion as formulated above, we observe that the reformulation of the problem is same as the original problem and hence the isotopic lifting and isoduality of the field $F(a,+,\times) \to \hat{F}(\hat{a},+,\hat{\times})$ does not change the solution of the problem. As the result we get the following theorem.

Theorem 3.3. *All possible isonormed isoalgebras with multiplicative isounit over the field of the isoreals are the isoalgebras of dimension 1 (isoreals), 2 (isocomplex), 4 (isoquaternions), and 8 (isooctonians) and the classification persists under isoduality.*

Further, lifting of addition gives the third formulation which is pseudoisotopic type

$$+ \to \hat{+} = +\hat{K}, \quad 0 \to \hat{0} = -\hat{K}, \quad \hat{K} = K \times \hat{1} \quad (141)$$

under which (137), (138) can be written over the pseudoisofield $\hat{F}(\hat{a},\hat{+},\hat{\times})$ as

$$(\hat{a}_1^{\hat{2}} \hat{+} \hat{a}_2^{\hat{2}} \hat{+} \ldots \hat{+} \hat{a}_n^{\hat{2}}) \hat{\times} (\hat{b}_1^{\hat{2}} \hat{+} \hat{b}_2^{\hat{2}} \hat{+} \ldots \hat{+} \hat{b}_n^{\hat{2}}) = \hat{A}_1^{\hat{2}} \hat{+} \hat{A}_2^{\hat{2}} \hat{+} \ldots \hat{+} \hat{A}_n^{\hat{2}} \quad (142)$$

with

$$\hat{A}_k = \sum_{r,s} \hat{c}_{krs} \hat{\times} \hat{a}_r \hat{\times} \hat{b}_s = (\sum_{r,s} c_{krs} a_r b_s) \hat{1} = A_k \times \hat{1} \quad (143)$$

This can be written in the conventional operations as

$$[(a_1^2 + a_2^2 + \ldots + a_n^2)\hat{1} + (n-1)K\hat{1}]T$$
$$[(b_1^2 + b_2^2 + \ldots + b_n^2)\hat{1} + (n-1)K\hat{1}] =$$
$$= (A_1^2 + A_2^2 + \ldots + A_n^2)\hat{1} + (n-1)K\hat{1}, \quad \hat{A}_k = A_k \hat{1} \quad (144)$$

The solution to (144) of dimension other than 1,2,4,8 under the pseudoisofield $\hat{F}(\hat{a},\hat{+},\hat{\times})$ was envisaged by prof.Santilli as a conjecture under the loss of the needed axioms of a field, such as distributive laws.

It was found that the solution do exist, but under the loss of number of axioms of the original field, in addition to the loss of distributivity. We consider a representative example of "Hidden numbers" of dimension 3 as follows

$$(\hat{1}^{\hat{2}} + \hat{2}^{\hat{2}} + \hat{3}^{\hat{2}}) * (\hat{5}^{\hat{2}} + \hat{6}^{\hat{2}} + \hat{7}^{\hat{2}}) = \hat{1}2^{\hat{2}} + \hat{2}4^{\hat{2}} + \hat{3}0^{\hat{2}} \quad (145)$$

Note that also the condition on $\hat{A}_k$ is true, that the elements in the r.h.s can be written as the combinations of the elements on the l.h.s as

$$12 = 2\times 6, \quad 24 = 2\times 5 + 2\times 7, \quad 30 = 3\times 3 + 3\times 7. \quad (146)$$

Hence we can rewrite the problem as

$$[(1^2 + 2^2 + 3^2)\hat{1} + 2K]T[(5^2 + 6^2 + 7^2)\hat{1} + 2K\hat{1}] = (12^2 + 24^2 + 30^2)\hat{1} + 2K\hat{1} \quad (147)$$

which on simplification gives a quadratic equation in K as

$$4K^2 + 246K - 80 = 0 \quad (148)$$

with solution

$$K = 0.325\ldots \quad (149)$$

Thus the solution exists, but is not an integer. This implies the loss of closure under isoaddition for the case of integers. However, the closure can be regained if the original field is enlarged to include all real numbers. The issue whether such solutions do indeed form a pseudoisofield is open for the mathematicians.

As algebras of dimensions higher than 8 are not alternative [21], also, as this property persists under isotopies and pseudoisotopies, leads to the fact that formulations (137)and (142) are restricted to dimensions $n \le 8$.

Prof. Santilli ref.[1] identified following open problems with regards to the notion of isofields.

- Investigative study of "number with singular unit", i.e. isofields of class IV which are at the foundations of the isotopic studies of gravitational collapse.
- The study of isofields of characteristic $p \neq 0$, to see whether new fields and therefore new Lie-algebras are permitted by isotopies.

Author of this article has defined 'Iso-Galois fields' ref.[44] which are basically finite isofields essentially of nonzero characteristic. As predicted by Santilli these isofields have important applications in Cryptography, Genetics, Fractal geometry etc.

- The study of the integro-differential topology characterized by isofields with local differential structure and integral isounits.

3.2. Genonumbers and Their Isoduals

We have seen that the two degrees of freedom due to isotopic lifting of *addition* and *multiplication* give rise to isofields and pseudoisofields respectively. These fields are at the foundation of the *Lie-isotopic theory* [8, 9, 45].

Also, there exists a third degree of freedom caused by the *ordering* of the above operations which leads to further generalization of a field which is at the foundation of *Lie-admissible algebras* [8, 9, 18].

Given a field $F(a,+,\times)$ of ordinary numbers with generic elements $a,b,c\ldots$, with addition $a+b=b+a$ and multiplication $a\times b$, we can define the following.

Genoaddition: Addition of a to b from the left, denoted by $a +^{>} b$ and addition of b to a from the right denoted by $a^{<} + b$ are called *genoadditions*.

Genomultiplication: Multiplication of a times b from the left denoted by $a \times^{>} b$, and multiplication b times a from the right denoted by $a^{<} \times b$ are called *genomultiplications*.

It is worthwhile to note that ordering of multiplication is fully compatible with its basic axioms, such as commutativity for real and complex numbers, associativity for quaternions, and alternativity for the octonions. In the case of real and complex numbers we will have

$$a \times^{>} b \equiv b \times^{>} a, \quad a^{<} \times b \equiv b^{<} \times a \quad (150)$$

The identity of multiplication from left and right can be different and hence two genomultiplications can very well be different i.e.

$$a \times^{>} b \neq a^{<} \times b \quad (151)$$

with realization,

$$a\times^{>} b := aRb, \quad a^{<}\times b := aSb, \quad R \neq S, \tag{152}$$

where R and S are fixed isotopic elements, called the genotopic elements. These are sufficiently smooth, bounded and nowhere singular (not necessarily Hermitean) outside the original field.

The left and right generalized *genounits* can be defined in the following manner

$$a\,\hat{\times}\,b \tag{153}$$

$$\hat{1} = T^{-1} \tag{154}$$

Note that all the axioms and properties of the original field are preserved under the mentioned left or right multiplication and multiplicative units under the appropriate ordering for all the dimensions 1,2,4,8. This procedure leads to new fields called as *genofield* denoted by $\hat{F}^{>}(\hat{a}^{>},+,\hat{\times}^{>})$ (right genofield) or ${}^{<}\hat{F}({}^{<}\hat{a},+,{}^{<}\hat{\times})$ (left genofield) or ${}^{<}\hat{F}^{>}({}^{<}\hat{a}^{>},+,{}^{<}\hat{\times}^{>})$. Also, *isodual genofields* are defined by the antiautomorphic conjugations

$$R \to R^{d} = -R, \quad S \to S^{d} = -S \tag{155}$$

denoted by ${}^{<}\hat{F}^{>d}({}^{<}\hat{a}^{>d},+,{}^{<}\hat{\times}^{>d})$.

Note that isofields are the particular case of genofields where the genotopic elements coincide. i.e.

$${}^{<}\hat{F}^{>d}({}^{<}\hat{a}^{>d},+,{}^{<}\hat{\times}^{>d})_{|R=S=T} = F(\hat{a},+,\hat{\times}). \tag{156}$$

R-S mutation of the Lie product: is defined as

$$(A,B) = ARB - BSA \tag{157}$$

which is Lie-admissible via the attached antisymmetric product

$$[A,\hat{B}] = (A,B) - (B,A) = ATB - BTA, T = R - S \tag{158}$$

which is Lie-isotopic.

The lifting $[A,B] \to [A,\hat{B}]$ is called an *isotopy*. The lifting $[A,B] \to (A,B)$ is called a *genotopy*, ref. [8, 1].

The Lie-isotopic algebras are defined by one single isotopy of the enveloping associative algebra and related unit

$$AB = A\times B \to A\,\hat{\times}\,B = ATB, \quad 1 \to \hat{1} = T^{-1}. \tag{159}$$

For the consistent formulation of Lie-isotopic algebras they must be defined over an isofield $\hat{F}(\hat{a},+,\hat{\times})$ with isounit $\hat{1} = T^{-1}$.

Note that for the conventional multiplication $\times$ there is no ordering as $1^{>} = {}^{<} 1 = 1$. The above ordering can be defined for isomultiplication $\hat{\times}$ wherein we can have different isounits.

The Lie-admissible algebras can be generated by two different isotopies of the original associative algebra using left and right isounits with corresponding isotopies as

$$AB \to ARB := A\times^{>} B, \quad 1 \to \hat{1}^{>} = R^{-1}, \tag{160}$$

$$BA \to BSA := B^{<}\times A, \quad 1 \to {}^{<}\hat{1} = S^{-1}. \tag{161}$$

which must be defined over the genofields ${}^{<}\hat{F}^{>}({}^{<}\hat{a}^{>},+,{}^{<}\hat{\times}^{>})$ with isounits ${}^{<}\hat{1}^{>}$. Here, the isounits related with the left and right isomultiplication are disjoint and can indeed be Hermitean and real-valued, which admit Kadeisville classification into classes I. II, III, IV and V.

However, in physics the isounits (left and right) used have a real physical significance when they are inter-related by a Hermitean conjugation as

$$\hat{1}^{>} = ({}^{<}\hat{1})^{\dagger} \tag{162}$$

This representation of the genounits (and hence genofields) provides approximation of irreversibility ref.[18].

It is important to note that conventional addition admits no meaningful ordering as $0^{>} = {}^{<} 0 \equiv 0$. However, the ordering exists for the isoaddition $\hat{+} = +K+$ as $\hat{+}^{>} \neq {}^{<}\hat{+}$ with $K^{>} \neq {}^{<} K$. But there is loss of distributive law for the resulting genofield under genoadditions ${}^{<}\hat{+}^{>}$.

All the above discussion leads to a broadest generalization of the existing theory of numbers through

1. *pseudogenofields* ${}^{<}\hat{F}^{>}({}^{<}\hat{a}^{>},{}^{<}\hat{+}^{>},{}^{<}\hat{\times}^{>})$ defined via genotopies of all aspects of conventional fields $F(a,+,\times)$ and
2. *isodual pseudogenofields* ${}^{<}\hat{F}^{>d}({}^{<}\hat{a}^{>d},{}^{<}\hat{+}^{>d},{}^{<}\hat{\times}^{>d})$ defined via isoduality of pseudogenofields.

This new generalization of the conventional numbers leads to the following categorization of numbers:

- Conventional numbers of dimension 1,2,4,8 and their isoduals;
- Isonumbers of the same dimension and their isoduals;
- Genonumbers of the same dimensions and their isoduals;
- Pseudoisonumbers of the same dimension and their isoduals;
- Pseudogenonumbers of the same dimension and their isoduals;
- "Hidden pseudoisonumbers" of dimension 3,4, 5,7 and their osoduals;
- "Hidden pseudogenonumbers" of dimension 3,4,5,7 and their isoduals.

Note that each of these can be defined for the fields of characteristic 0 or for $p \neq 0$.

In addition to above generalization, we can have an ordered set of values for the multiplicative unit such as

$1^{>} = \{2, \frac{4}{3}, 6, \ldots\}$ defined as applicable or to the right or left.

This possibility leads to the new numbers called as *hyper-Santillian numbers*. These include hyper-real, hyper complex, hyper-quaternion numbers which have vast applications in biological sciences.

In the further generalization, the multiplicative unit can very well have non-zero negative values. This leads to a new class of numbers called *iso-dual Santillian numbers*. This further leads to a new kinds of conventional iso-dual numbers called as *iso-topic isodual numbers, geno-topic iso-dual numbers* and *hyper-structural isodual*. These numbers have applications for antimatter.

The above generalization of the conventional numbers gives us, in all, eleven classes of new numbers namely, *the iso-topic numbers, genotopic to the right and left, right and left hyper-structural numbers, iso-dual conventional numbers, iso-dual iso-topic numbers, iso-dual geno-topic to the right and left numbers and hyper-structural iso-dual to the right and left numbers.* Each class is applicable to the real, complex and quaternion numbers where each of the applications have infinite number of possible units.

4. Applications and Advances

Quantum mechanics was sufficient to deal with 'Exterior Dynamical systems' which are liner, local, lagrangian and hamiltonian. The main purpose of formulating the new generalized mathematics was to deal with the insufficiencies in the modern mathematics to describe 'Interior Dynamical systems' which are intrinsically non-linear, non-local, non-hamiltonian and non-lagrangian. The axiom-preserving generalization of quantum mechanics which can also deal with non-linear, non-local non-hamiltonian and non-lagrangian systems is called the *Hadronic mechanics*. The mechanics; built specifically to deal with 'hadrons' (strongly interacting particles) ref. [18]. Prof. Santilli, in 1978 when at Harvard University, proposed 'Hadronic mechanics' under the support from U. S. Department of Energy, which was subsequently studied by number of mathematicians, theoreticians and experimentalists. Hadronic mechanics is directly universal; that is, capable of representing all possible nonlinear, nonlocal, nonhamiltonian, continuous or discrete, inhomogeneous and anisotropic systems (universality), directly in the frame of the experimenter (direct universality). In particular the hadronic mechanics has shown that quantum mechanics is completely inapplicable to the synthesis of neutron [46], as mass of the neutron is greater than the sum of the masses of proton and electron (called "mass defect") of which it is made. In this case quantum equations are completely inconsistent. Hadronic mechanics has achieved numerically exact results in the cases in which quantum mechanics results are not valid. For further details of isonumber theory we recommend refs. [47, 1, 48, 46, 49].

As far as mathematics is concerned, one of the major applications of isonumber theory is in Cryptography, ref. [50]. Cryptograms can be lifted to iso-cryptograms which render highest security for a given crypto-system. Isonumbers, hypernumbers and their pseudo-formulations can be used effectively for the tightest security via new disciplines, *isocryptology, genocryptology, hypercryptology, pseudocryptology* etc. More complex cryptograms can be achieved using *pseudocryptograms* in which we have the additional hidden selection of addition and multiplication to the left and those to the right whose results are generally different among themselves. Yet more complex pseudocryptograms can be achieved in which the result of each individual operations of addition and multiplication is given by a set of numbers [50]. *Santillian iso-crypto systems* have maximum security due to a large variety of isounits which can be changed automatically and continuously, achieving maximum possible security needed for the modern age banking and other systems related with information technology.

Reformulations of conventional numbers to the most generalized isonumbers and subsequently to genonumbers and hypernumbers led to a vast variety of parallel developments in the conventional mathematics including hyperstructures [51] and its various branches such as 'iso-functional analysis' ref [35], iso-calculus ref [52], iso-cryptography [50] etc.

Iso-Galois fields [53], Iso-permutation groups [54, 53] have been defined by this author, which can play an important role in cryptography and other branches of mathematics where finite fields are used. Investigations are underway.

Isomathematics can also explain complex biological structures and hence has applications in Fractal geometry. Further applications in Neuroscience and Genetics can provide new insight in these disciplines.

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