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On the Practical Exponential Stability in Mean Square of Stochastic Perturbed Systems Via a Lyapunov Approach

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Abstract: The Lyapunov method is one of the most effective methods to analyze the stability of stochastic differential equations (SDEs). Different authors analyzed the stability of SDEs based on Lyapunov techniques when the origin can be considered as an equilibrium point. When the origin is not necessarily an equilibrium point, it is still possible to analyze the asymptotic stability of solutions concerning a small neighborhood of the origin. The purpose is to study the asymptotic stability of a system whose solution behavior is a small ball of state space or close to it. Thus, all state trajectories are bounded and close to a sufficiently small neighborhood of the origin. In this sense, the limited boundedness of solutions of random systems, or the chance of convergence of solutions needs to be analyzed on a ball centered on the origin. This is the so called "Practical Stability". In this article, we mainly investigate the practical uniform exponential stability in the mean square of stochastic linear time–invariant systems. In addition, we are developing the problem of stabilization of certain classes of perturbed stochastic systems. Our crucial techniques include Lyapunov techniques and generalized Gronwall inequalities. Lastly, we provide a numerical example to illustrate our theoretical findings.

Keywords: Linear Systems, Stochastic Systems, Lyapunov Techniques, Brownian Motions, Nontrivial Solution, Practical Stability, Stabilization

1. Introduction

Stochastic differential equations are beneficial for the modelling of physical, technical, biological and economic systems in which there is significant uncertainty.

In fact, noise can destabilize a given stable system. Nevertheless, it can further stabilise a given unstable system or render a given stable system even more stable, we mention here ([4], [7], [11], [14], [15]) among others.

As it is for deterministic systems, Lyapunov function is a powerful tool for qualitative analysis of stochastic differential equations, without knowing the explicit solution form.

To investigate the stability of stochastic differential equations, we generally look at solutions in the neighborhood of the origin, which is regarded as an equilibrium point. Thus, In this way, we can study the stability of the solutions in relation to a small neighborhood of origin. when origin is not always an equilibrium point, almost all state paths are delineated and approaching the origin in a sufficiently small neighborhood, that's what we call it "Practical Stability". The concept of practical stability was developed by different authors in ([1], [3], [16], [17]) and references therein.

Actually, it is very important and very useful for stability analysis or for the design of practical dynamical controllers systems given that controlling the system to an idealized point is costly or impossible in the presence of disturbances and the best that we can hope for in these situations is practical stability.

In practical terms, it may be sufficient to stabilize a system in the phase space region where implementation is still acceptable. It is common knowledge that asymptotic stability is more important than stability. Even the selected system can oscillate close to the source. For this reason, the concept of practical stability is more appropriate in various cases than asymptotic stability.

In this case, all state trajectories are limited and close to a sufficiently small neighborhood of origin. It is also hoped that the State will approach the origin (or certain sufficiently small neighborhoods) in a sufficiently rapid manner, especially in the presence of disturbances. Generally, we know some information about the upper limit of the disturbance term whose size affects the size of the bullet.

Recently, the concept of practical stability of nonlinear stochastic differential equations is examined through the techniques of Lyapunov, see ([5], [6]) and references therein.

Motivated by the above discussion, in this document our central objective is to define and develop the global uniform exponential stability practical in the mean square for a class of time-invariant linear systems under Brownian motion disturbances in the Lyapunov approach.

The remainder of this paper has the following structure: In Section 2, we set up sufficient conditions for the global practical uniform exponential stability in mean square of linear time–invariant stochastic perturbed systems by using Lyapunov techniques and non–linear generalized integral inequalities. In Section 3, we give sufficient conditions ensuring the practical uniform exponential stabilization in mean square of stochastic systems by using Lyapunov techniques and Itô's formula. In Section 4, we present an example illustrating the efficiency of the results achieved. Finally, we wrap up the document with a conclusion.

Notations:

Let $\mathcal{R}^d$ represent the real d-dimensional space; $\mathcal{R}_+$ stands for all the non–negative numbers set; $\mathcal{R}^{d\times n}$ denotes the real $d \times n$ matrix space. For a vector χ, let $||\chi||$ denotes its usual Euclidean norm, χ^{T} be the transpose; For the matrix $\mathcal{F}$, $||\mathcal{F}|| = \sqrt{\lambda_{\max}(\mathcal{M}^{\mathrm{T}}\mathcal{F})}$ refers to its Euclidean matrix standard, where $\mathrm{Tr}(\cdot)$ is the matrix trace, $\lambda_{\min}(\mathcal{F})$ and $\lambda_{\max}(\mathcal{F})$ indicate the minimum and maximum eigenvalue of $\mathcal{F}$, respectively; $\mathbb{I}$ is Identity Matrix.

2. Practical Stability in Mean Square

In this section, we study the exponential stability in the mean square of linear time–invariant stochastic perturbed systems when the origin is not an equilibrium point. We propose the concept of practical exponential stability in mean square and we provide sufficient conditions for such stability.

Consider this linear time–invariant system:

$$\dot{z}(\mathrm{t}) = \mathcal{M}z(\mathrm{t}), \quad z(\mathrm{t}_0) = z_0, \tag{1}$$

where $z \in \mathcal{R}^d$ is a system status vector, $z(\mathrm{t}_0) = z_0 \in \mathcal{R}^d$ is the initial condition, $\mathcal{M}$ is a constant matrix $(d \times d)$.

Let us assume that certain parameters of linear time–invariant system (1) are excited or disturbed by certain environmental sounds (for further details see Oksendal [12], Mao [10]). Then, we get the next linear time–invariant perturbed stochastic system.

$$dz(\mathrm{t}) = \mathcal{M}z(\mathrm{t})d\mathrm{t} + \mathrm{F}(\mathrm{t}, z(\mathrm{t}))d\mathcal{B}_{\mathrm{t}}, \tag{2}$$

where $z(\mathrm{t}_0) = z_0 \in \mathcal{R}^d$ is the initial system state, $\mathrm{F} : \mathcal{R}_+ \times \mathcal{R}^d \longrightarrow \mathcal{R}^{d\times n}$, $\mathcal{B}_t = (\mathcal{B}_1(\mathrm{t}), \cdots, \mathcal{B}_n(\mathrm{t}))^{\mathrm{T}}$ is an n–dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We assume that the function $\mathrm{F}(\mathrm{t}, z)$ satisfies the following relations $\forall \mathrm{t} \geq 0,\ \forall z \in \mathcal{R}^d$ and $\forall \widetilde{z} \in \mathcal{R}^d$,

$$||\mathrm{F}(\mathrm{t}, z)||^2 \leq \tau_1(1 + ||z||^2), \tag{3}$$

$$||\mathrm{F}(\mathrm{t}, z) - \mathrm{F}(\mathrm{t}, \widetilde{z})|| \leq \tau_2||z - \widetilde{z}||, \tag{4}$$

where τ_1 and τ_2 are two given strictly positive constants.

Then, there is a unique global $z(\mathrm{t}, \mathrm{t}_0, z_0)$ solution for the initial condition corresponding to the initial condition $z_0 \in \mathcal{R}^d$ (see Mao [10], Oksendal [12] for additional details). In the following we use $z(\mathrm{t}, \mathrm{t}_0, z_0)$, or simply $z(\mathrm{t})$ denote a solution of our system over a short interval of the perturbed stochastic system (2).

We assume that there exists t such that $\mathrm{F}(\mathrm{t}, 0) \neq 0$. As a result, the origin is no longer an equilibrium point for linear time–invariant stochastic perturbed system (2).

We will investigate the practical exponential stability in mean square of the solutions of the linear time–invariant stochastic perturbed system when 0 is not an equilibrium point, but in a small neighborhood of the origin in terms of convergence of solution in probability to a small ball:

$$\mathbb{B}_r := \left\{z \in \mathcal{R}^d : \|z\| \leq r\right\},\ r > 0.$$

Definition 2.1

1. The ball $\mathbb{B}_r$ is globally uniformly exponentially stable in mean square, if there exist two positive constants κ_1 and κ_2, such that for all $\mathrm{t}_0 \in \mathcal{R}_+$, and all $z_0 \in \mathcal{R}^d$, the following inequality is

$$\mathrm{E}(||z(\mathrm{t})||^2) \leq \kappa_1||z_0||^2 e^{-\kappa_2(\mathrm{t}-\mathrm{t}_0)} + r, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

2. The linear time–invariant stochastic perturbed system (2) is said to be globally practically uniformly exponentially stable in mean square, if there exists $r > 0$, such that the ball $\mathbb{B}_r$ is globally uniformly exponentially stable in mean square.

Theorem 2.1 Let $z(\mathrm{t})$ be an n–dimensional Itô process on $t \geq 0$ performs the following stochastic differential equation:

$$dz(\mathrm{t}) = a(\mathrm{t}, z(\mathrm{t}))d\mathrm{t} + b(\mathrm{t}, z(\mathrm{t}))d\mathcal{B}_t,$$

where $a \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R}^d)$ and $b \in \mathcal{L}^2(\mathcal{R}_+, \mathcal{R}^{d\times n})$.

Let's define $\mathcal{V} \in \mathcal{C}^{1.2}(\mathcal{R}_+ \times \mathcal{R}^d, \mathcal{R})$: the family of all non–negative functions $\mathcal{V}(\mathrm{t}, z(\mathrm{t}))$ defined on $\mathcal{R}_+ \times \mathcal{R}^d$ which are once continuously differentiable with respect to t and twice with respect to z. Hence, $\mathcal{V}(\mathrm{t}, z(\mathrm{t}))$ is an ltô process and

$$d\mathcal{V}(\mathrm{t}, z(\mathrm{t})) = \mathcal{L}\mathcal{V}(\mathrm{t}, z(\mathrm{t}))d\mathrm{t} + \mathcal{V}_z(\mathrm{t}, z(\mathrm{t}))b(\mathrm{t}, z(\mathrm{t}))d\mathcal{B}_{\mathrm{t}},$$

where

$$\mathcal{L}\mathcal{V}(\mathrm{t},z) := \mathcal{V}_t(\mathrm{t},z) + \mathcal{V}_z(\mathrm{t},z)a(\mathrm{t},z) + \frac{1}{2}\mathrm{trace}\left(\mathrm{b}^{\mathrm{T}}(\mathrm{t},z)\mathcal{V}_{zz}(\mathrm{t},z)b(\mathrm{t},z)\right),$$

$$\mathcal{V}_t(\mathrm{t},z) = \frac{\partial \mathcal{V}}{\partial t}(\mathrm{t},z)\ ,\ \mathcal{V}_z(\mathrm{t},z) = \left(\frac{\partial \mathcal{V}}{\partial z_1}(\mathrm{t},z), \frac{\partial \mathcal{V}}{\partial z_2}(\mathrm{t},z), \cdots, \frac{\partial \mathcal{V}}{\partial z_d}(\mathrm{t},z)\right)\ ,\ \mathcal{V}_{zz}(\mathrm{t},z) = \left(\frac{\partial^2 \mathcal{V}}{\partial z_i \partial z_j}(\mathrm{t},z)\right)_{d\times d}.$$

Remark 2.2 Different authors tackle the problem of stability of linear time–invariant systems within Lyapunov techniques, see [2, 9]. In fact, one seeks solutions $\mathscr{P} \in \mathcal{R}^{d\times d}$ as well as $\mathscr{Q} \in \mathcal{R}^{d\times d}$ of the ensuing Lyapunov equation:

$$\mathcal{M}^{\mathrm{T}}\mathscr{P} + \mathscr{P}\mathcal{M} = -\mathscr{Q}, \tag{5}$$

and the related Lyapunov function candidate is as follows:

$$\mathcal{V} : \mathcal{R}^d \to \mathcal{R}_+, \quad y \longmapsto z^{\mathrm{T}}\mathscr{P}z. \tag{6}$$

Remark 2.3 Our strategy consists of using the Lyapunov function (6) for deterministic linear time–invariant system (1) as Lyapunov function for perturbed stochastic system (2) under certain constraints on the perturbation function.

Theorem 2.2 Consider linear time–invariant perturbed stochastic system (2), if there is a positive defined symmetrical matrix $\mathscr{P}$, being the solution of the Lyapunov matrix equation (6), with $\mathscr{Q}$ is a positive defined symmetric matrix. In addition, we assume that the perturbation term $\mathrm{F}(\mathrm{t},z)$ satisfies for all $t \geq 0$, and all $z \in \mathbb{R}^d$, the following assumption:

$$||\mathrm{F}(\mathrm{t},z)||^2 \leq \Gamma||z||^2 + \Psi(\mathrm{t}), \quad \text{a.s.} \tag{7}$$

where Γ is a positive constant and Ψ is a non–negative bounded continuous function. Then, the stochastic system (2) is practically uniformly exponentially stable in mean square.

The next generalized Gronwall lemma is essential for proving our theorem. *Theorem 2.3* Let $f : \mathcal{R}_+ \longrightarrow \mathcal{R}_+$ be a continuous function, ϵ is a positive real number and u is a strict positive real constant. Suppose that for all $\mathrm{t} \in \mathcal{R}_+$, and $0 \leq u \leq \mathrm{t}$, we have

$$f(\mathrm{t}) - f(u) \leq \int_u^{\mathrm{t}} (-\lambda f(\mathrm{s}) + \epsilon) d\mathrm{s}.$$

Then, we obtain

$$f(\mathrm{t}) \leq \frac{\epsilon}{\lambda} + f(0)\exp(-\lambda \mathrm{t}).$$

Remark 2.6 If we replace $f(0)$ by $f(\mathrm{t}_0)$ with $0 \leq \mathrm{t}_0 \leq u \leq \mathrm{t}$, this last lemma is still true.

RProof of Theorem 2.4. Through the generalized Itô's formula 2.1 applied at $\mathcal{V}(z(\cdot))$ where $z(\cdot)$ the trajectory of the stochastic system (2), one can derive that for $\mathrm{t} \geq 0$,

$$\begin{aligned}
\mathcal{L}\mathcal{V}(z(\mathrm{t})) &= z^{\mathrm{T}}(\mathrm{t})(\mathcal{M}^{\mathrm{T}}\mathscr{P} + \mathscr{P}\mathcal{M})z(\mathrm{t}) + \mathrm{F}^{\mathrm{T}}(\mathrm{t},z(\mathrm{t}))\mathscr{P}\mathrm{F}(\mathrm{t},z(\mathrm{t})),\\
&= -z^{\mathrm{T}}(\mathrm{t})\mathscr{Q}z(\mathrm{t}) + \mathrm{F}^{\mathrm{T}}(\mathrm{t},z(\mathrm{t}))\mathscr{P}\mathrm{F}(\mathrm{t},z(\mathrm{t})),\\
&\leq -z^{\mathrm{T}}(\mathrm{t})\mathscr{Q}z(\mathrm{t}) + ||\mathscr{P}||\ ||\mathrm{F}(\mathrm{t},z(\mathrm{t}))||^2,\\
&\leq -z^{\mathrm{T}}(\mathrm{t})\mathscr{Q}z(\mathrm{t}) + \lambda_{\max}(\mathscr{P})||\mathrm{F}(\mathrm{t},z(\mathrm{t}))||^2.
\end{aligned}$$

By virtue (7), it follows

$$\begin{aligned}
\mathcal{L}\mathcal{V}(z(\mathrm{t})) &\leq -\lambda_{\min}(\mathscr{Q})z^{\mathrm{T}}(\mathrm{t})z(\mathrm{t}) + \Gamma\lambda_{\max}(\mathscr{P})z^{\mathrm{T}}(\mathrm{t})z(\mathrm{t}) + \lambda_{\max}(\mathscr{P})\Psi(\mathrm{t}),\\
&= -\left(\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})\right)z^{\mathrm{T}}(\mathrm{t})z(\mathrm{t}) + \lambda_{\max}(\mathscr{P})\Psi(\mathrm{t}),
\end{aligned}$$

without losing generality, we may assume that

$$\lambda_{\min}(\mathscr{Q}) > \Gamma\lambda_{\max}(\mathscr{P}).$$

Accordingly, we obtain

$$\begin{aligned}
\mathcal{L}\mathcal{V}(z(\mathrm{t})) &\leq -\frac{\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})}{\lambda_{\max}(\mathscr{P})}z^{\mathrm{T}}(\mathrm{t})\mathscr{P}z(\mathrm{t}) + \lambda_{\max}(\mathscr{P})\Psi(\mathrm{t}),\\
&= -\left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right)\mathcal{V}(z(\mathrm{t})) + \lambda_{\max}(\mathscr{P})\Psi(\mathrm{t}).
\end{aligned}$$

Noting that the function $\mathrm{t} \mapsto \Psi(\mathrm{t})$ is a non–negative bounded on $[0,\infty)$, so there exist $m > 0$, such that $\Psi(\mathrm{t}) \leq m$. Then, we arrive at

$$\mathcal{L}\mathcal{V}(z(\mathrm{t})) \leq \left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right)\mathcal{V}(z(\mathrm{t})) + m\lambda_{\max}(\mathscr{P}).$$

Applying Dynkin's formula [8], we derive

$$\mathrm{E}(\mathcal{V}(z(\mathrm{t}))) - \mathcal{V}(z(0)) = \int_{\mathrm{t}_0}^{\mathrm{t}} \mathrm{E}(\mathcal{L}\mathcal{V}(z(\mathrm{s})))d\mathrm{s}.$$

Accordingly, for all u, t with $0 \leq \mathrm{t}_0 \leq u \leq \mathrm{t} \leq \infty$, we see

$$\begin{aligned} 0 \leq \mathrm{E}(\mathcal{V}(z(\mathrm{t}))) - \mathrm{E}(\mathcal{V}(z(u))) &\leq \int_u^{\mathrm{t}} \mathrm{E}(\mathcal{V}(z(\mathrm{s})))ds, \\ &\leq \int_u^{\mathrm{t}} -\left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right) \mathrm{E}(\mathcal{V}(z(\mathrm{s}))) + m\lambda_{\max}(\mathscr{P})d\mathrm{s}. \end{aligned}$$

Applying the generalized Gronwall lemma (lemma 2.5), one deduce that

$$\mathrm{E}(\mathcal{V}(z(\mathrm{t}))) \leq \mathrm{E}(\mathcal{V}(z(\mathrm{t}_0))) \exp\left(-\left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right)(\mathrm{t}-\mathrm{t}_0)\right) + \frac{m\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})}.$$

Next, we need to infer an estimation for the expectation of the norm of $z(\cdot)$.

$$\begin{aligned} \mathrm{E}(||z(\mathrm{t})||^2) &\leq \frac{1}{\lambda_{\min}(\mathscr{P})}\mathrm{E}\left(z^{\mathrm{T}}(\mathrm{t})\mathscr{P}z(\mathrm{t})\right), \\ &\leq \frac{1}{\lambda_{\min}(\mathscr{P})}\mathrm{E}(\mathcal{V}(z(\mathrm{t}_0))) \exp\left(-\left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right)(\mathrm{t}-\mathrm{t}_0)\right) + \frac{m\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})\left(\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})\right)}, \\ &= \frac{1}{\lambda_{\min}(\mathscr{P})} z^{\mathrm{T}}(\mathrm{t}_0)\mathscr{P}z(\mathrm{t}_0) \exp\left(-\left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right)(\mathrm{t}-\mathrm{t}_0)\right) + \frac{m\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})\left(\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})\right)}, \\ &\leq \frac{\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})}||z_0||^2 \exp\left(-\left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right)(\mathrm{t}-\mathrm{t}_0)\right) + \frac{m\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})\left(\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})\right)}. \end{aligned}$$

Next, this follows for all $\mathrm{t} \geq \mathrm{t}_0 \geq 0$, and all $z_0 \in \mathcal{R}^d$,

$$\mathrm{E}(||z(\mathrm{t})||^2) \leq \frac{\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})}||z_0||^2 \exp\left(-\left(\frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma\right)(\mathrm{t}-\mathrm{t}_0)\right), + \frac{m\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})\left(\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})\right)}.$$

For $\kappa_1 = \frac{\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})}$, $\kappa_2 = \frac{\lambda_{\min}(\mathscr{Q})}{\lambda_{\max}(\mathscr{P})} - \Gamma$, and $r = \frac{m\lambda_{\max}(\mathscr{P})}{\lambda_{\min}(\mathscr{P})\left(\lambda_{\min}(\mathscr{Q}) - \Gamma\lambda_{\max}(\mathscr{P})\right)}$ the linear time–invariant stochastic perturbed system (2) is globally practically uniformly exponentially stable in mean square. □

3. Stabilization Problem

In this section, we develop the stabilization of certain classes of stochastic perturbed systems via Lyapunov techniques.

Consider the next stochastic system:

$$\begin{cases} dz(\mathrm{t}) = \left(\widetilde{\mathcal{M}}z(\mathrm{t}) + \mathcal{N}U\right)d\mathrm{t} + \mathrm{H}(\mathrm{t}, z(\mathrm{t}))d\mathcal{B}_{\mathrm{t}} \\ z(\mathrm{t}_0) = z_0, \end{cases} \tag{8}$$

where $z(\mathrm{t}) \in \mathcal{R}^d$ is the system state vector, $\mathrm{U} \in \mathcal{R}^d$ is the control input, $\mathcal{B}_t = (\mathcal{B}_1(\mathrm{t}), \cdots, \mathcal{B}_n(\mathrm{t}))$ is an n–Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\widetilde{\mathcal{M}} \in \mathcal{R}^{d\times d}$ is a constant matrix, $\mathcal{N} \in \mathcal{R}^{n\times d}$ is a constant matrix, and function $\mathrm{H} : \mathcal{R}_+ \times \mathcal{R}^d \to \mathcal{R}^{d\times n}$ satisfies both conditions (3) and (4).

Definition 3.1 The stochastic system (8) is said to be practically uniformly exponentially stabilizable in mean square, if there exists a state feedback control law:

$$\mathrm{U} = \mathrm{U}(z),$$

such that the following closed–loop stochastic system:

$$\left(\widetilde{\mathcal{M}}z(\mathrm{t}) + \mathcal{N}\mathrm{U}(z)\right)d\mathrm{t} + \mathrm{H}(\mathrm{t}, z(\mathrm{t}))d\mathcal{B}_{\mathrm{t}}, \tag{9}$$

is practically uniformly exponentially stable in mean square.

We define feedback law in the following way:

$$\mathrm{U}(z) = \mathcal{D}z,$$

that stabilizes the linear part, where $\mathcal{D} \in \mathcal{R}^{n\times d}$ is a constant matrix.

We will set out a few assumptions that we will require later on:

$(\mathcal{H}_1)$ There exists a positive definite symmetric matrix $\widetilde{\mathscr{P}}$, resolve the following Lyapunov matrix equation:

$$(\widetilde{\mathcal{M}} + \mathcal{N}\mathcal{D})^{\mathrm{T}}\widetilde{\mathscr{P}} + \widetilde{\mathscr{P}}(\widetilde{\mathcal{M}} + \mathcal{N}\mathcal{D}) = -\widetilde{\mathscr{Q}},$$

where $\widetilde{\mathscr{Q}}$ is a positive definite symmetric matrix.

$(\mathcal{H}_2)$ There exists a continuous non–negative bounded function, such that

$$||\mathrm{H}(\mathrm{t}, z)|| \leq \rho(\mathrm{t})||z||, \quad \forall(\mathrm{t}, z) \in \mathcal{R}_+ \times \mathcal{R}^d.$$

Theorem 3.1 Under assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, the closed–loop stochastic system (9) is practically uniformly exponentially stable in mean square.

Proof. Consider Lyapunov's next function:

$$\mathcal{V}(z) = z^{\mathrm{T}}\widetilde{\mathscr{P}}z,$$

By the generalized Itô formula 2.1 to $\mathcal{V}(z(\cdot))$ where $z(\cdot)$ is a trajectory of the closed–loop stochastic system (9), we obtain

$$\mathcal{LV}(z(\mathrm{t})) = z^{\mathrm{T}}(\mathrm{t})\left((\widetilde{\mathcal{M}}+\mathcal{ND})^{\mathrm{T}}\widetilde{\mathscr{P}}+\widetilde{\mathscr{P}}(\widetilde{\mathcal{M}}+\mathcal{ND})\right)z(\mathrm{t})+\mathrm{H}^{\mathrm{T}}(\mathrm{t},z(\mathrm{t}))\widetilde{\mathscr{P}}\mathrm{H}(\mathrm{t},z(\mathrm{t})).$$

Using Assumption $(\mathcal{H}_1)$, it follows that

$$\mathcal{LV}(z(\mathrm{t})) \leq -z^{\mathrm{T}}(\mathrm{t})\widetilde{\mathscr{Q}}z(\mathrm{t})+||\widetilde{\mathscr{P}}||||\mathrm{F}(\mathrm{t},z(\mathrm{t}))||^2.$$

By $(\mathcal{H}_2)$, we arrive at

$$\mathcal{LV}(z(\mathrm{t})) \leq -z^{\mathrm{T}}(\mathrm{t})\widetilde{\mathscr{Q}}z(\mathrm{t})+\lambda_{max}(\widetilde{\mathscr{P}})\rho^2(\mathrm{t}).$$

The function $\rho(\mathrm{t})$ is continuous non–negative bounded, then there exists $\widetilde{\rho}>0$, such that

$$||\rho(\mathrm{t})|| \leq \widetilde{\rho}, \quad \forall \mathrm{t} \geq \mathrm{t}_0 \geq 0.$$

Therefore, it derives that

$$\begin{aligned}\mathcal{LV}(z(\mathrm{t})) &\leq -z^{\mathrm{T}}(\mathrm{t})\widetilde{\mathscr{Q}}z(\mathrm{t})+\lambda_{max}(\widetilde{\mathscr{P}})\widetilde{\rho}^2,\\ &\leq -\lambda_{min}(\widetilde{\mathscr{Q}})z^{\mathrm{T}}(\mathrm{t})z(\mathrm{t})+\lambda_{max}(\widetilde{\mathscr{P}})\widetilde{\rho}^2.\end{aligned}$$

Accordingly, we obtain

$$\mathcal{LV}(z(\mathrm{t})) \leq -\frac{\lambda_{\min}(\widetilde{\mathscr{Q}})}{\lambda_{max}(\widetilde{\mathscr{P}})}z^{\mathrm{T}}(\mathrm{t})\widetilde{\mathscr{P}}z(\mathrm{t})+\widetilde{\rho}^2.$$

By using Dynkin's formula [8], we arrive at

$$\mathrm{E}(\mathcal{V}(z(\mathrm{t})))-\mathcal{V}(z(0)) = \int_0^{\mathrm{t}}\mathrm{E}(\mathcal{LV}((z(\mathrm{s})))d\mathrm{s},$$

which implies that $\forall u,\ \mathrm{t},\ 0 \leq \mathrm{t}_0 \leq u \leq \mathrm{t} \leq \infty$,

$$0 \leq \mathrm{E}(\mathcal{V}(z(\mathrm{t})))-\mathrm{E}(\mathcal{V}(z(u))) \leq \int_u^{\mathrm{t}}\mathrm{E}(\mathcal{V}(z(\mathrm{s}))d\mathrm{s}, \leq \int_u^{\mathrm{t}} -\frac{\lambda_{\min}(\widetilde{\mathscr{Q}})}{\lambda_{max}(\widetilde{\mathscr{P}})}\mathrm{E}(\mathcal{V}(z(\mathrm{s})))+\widetilde{\rho}^2 d\mathrm{s}.$$

By applying the Gronwall lemma (Lemma 2.5), we derive

$$\mathrm{E}(\mathcal{V}(z(\mathrm{t}))) \leq \mathrm{E}(\mathcal{V}(z(\mathrm{t}_0)))\exp\left(-\frac{\lambda_{\min}(\widetilde{\mathscr{Q}})}{\lambda_{max}(\widetilde{\mathscr{P}})}(\mathrm{t}-\mathrm{t}_0)\right)+\frac{\widetilde{\rho}^2\lambda_{\max}(\widetilde{\mathscr{P}})}{\lambda_{\min}(\widetilde{\mathscr{Q}})}.$$

Besides, we arrive at

$$\begin{aligned}\mathrm{E}(||z(\mathrm{t})||^2) &\leq \frac{1}{\lambda_{min}(\widetilde{\mathscr{P}})}\mathrm{E}(z^{\mathrm{T}}(\mathrm{t})\widetilde{\mathscr{P}}z(\mathrm{t})),\\ &\leq \frac{1}{\lambda_{\min}(\widetilde{\mathscr{P}})}\mathrm{E}(\mathcal{V}(z(\mathrm{t}_0)))\exp\left(-\frac{\lambda_{\min}(\widetilde{\mathscr{Q}})}{\lambda_{max}(\widetilde{\mathscr{P}})}(\mathrm{t}-\mathrm{t}_0)\right)+\frac{\widetilde{\rho}^2\lambda_{\max}(\widetilde{\mathscr{P}})}{\lambda_{\min}(\widetilde{\mathscr{Q}})\lambda_{\min}(\widetilde{\mathscr{P}})},\\ &= \frac{1}{\lambda_{\min}(\widetilde{\mathscr{P}})}z^{\mathrm{T}}(\mathrm{t}_0)\widetilde{\mathscr{P}}z(\mathrm{t}_0)\exp\left(-\frac{\lambda_{\min}(\widetilde{\mathscr{Q}})}{\lambda_{max}(\widetilde{\mathscr{P}})}(\mathrm{t}-\mathrm{t}_0)\right)+\frac{\widetilde{\rho}^2\lambda_{\max}(\widetilde{\mathscr{P}})}{\lambda_{\min}(\widetilde{\mathscr{Q}})\lambda_{\min}(\widetilde{\mathscr{P}})},\\ &\leq \frac{\lambda_{\max}(\widetilde{\mathscr{P}})}{\lambda_{\min}(\widetilde{\mathscr{P}})}||z_0||^2\exp\left(-\frac{\lambda_{\min}(\widetilde{\mathscr{Q}})}{\lambda_{max}(\widetilde{\mathscr{P}})}(\mathrm{t}-\mathrm{t}_0)\right)+\frac{\widetilde{\rho}^2\lambda_{\max}(\widetilde{\mathscr{P}})}{\lambda_{\min}(\widetilde{\mathscr{Q}})\lambda_{\min}(\widetilde{\mathscr{P}})}.\end{aligned}$$

Consequently, we see that for all $t \geq \mathrm{t}_0 \geq 0$, and all $z_0 \in \mathcal{R}^d$,

$$\mathrm{E}(||z(\mathrm{t})||^2) \leq \frac{\lambda_{\max}(\widetilde{\mathscr{P}})}{\lambda_{\min}(\widetilde{\mathscr{P}})}||z_0||^2\exp\left(-\frac{\lambda_{\min}(\widetilde{\mathscr{Q}})}{\lambda_{max}(\widetilde{\mathscr{P}})}(\mathrm{t}-\mathrm{t}_0)\right)+\frac{\widetilde{\rho}^2\lambda_{\max}(\widetilde{\mathscr{P}})}{\lambda_{\min}(\widetilde{\mathscr{Q}})\lambda_{\min}(\widetilde{\mathscr{P}})}.$$

That is, the closed–loop stochastic system (9) is practically uniformly exponentially stable in mean square. □

4. Example

In this section, we provide an example to demonstrate the validity of our findings. Let us look at the following stochastic system:

$$dz(\mathrm{t}) = \mathcal{M}z(\mathrm{t})d\mathrm{t} + \mathrm{F}(\mathrm{t}, z(\mathrm{t}))d\mathcal{B}_t, \qquad (10)$$

where $z = (z_1, z_2) \in \mathcal{R}^2$.

$$\mathcal{M} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathrm{F}(\mathrm{t}, z) = \begin{pmatrix} \mathrm{F}_1(\mathrm{t}, z) \\ \mathrm{F}_2(\mathrm{t}, z) \end{pmatrix},$$

with

$$\begin{cases} \mathrm{F}_1(\mathrm{t}, z) = \dfrac{1}{8}\dfrac{z_1^2}{1+\sqrt{z_1^2+z_2^2}} + \dfrac{1}{4}\sin(2\pi t), \\ \\ \mathrm{F}_2(\mathrm{t}, z) = \dfrac{1}{8}\dfrac{z_2^2}{1+\sqrt{z_1^2+z_2^2}} + \dfrac{1}{\sqrt{2}^3}e^{-t}, \end{cases}$$

with initial value $z_0 = (z_{1_0}, z_{2_0})$.

System (10) can be considered as a linear time–invariant stochastic perturbed system of:

$$dz(\mathrm{t}) = \mathcal{M}z(\mathrm{t})d\mathrm{t}.$$

We take:

$$\mathscr{P} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \quad \mathscr{Q} = \mathbb{I}.$$

It is easily observed that $\mathscr{P}$ and $\mathscr{Q}$ are both definite positive symmetric matrices hat resolve the Lyapunov equation Eq.(6).

By using Matlab we get:

$$\lambda_{\min}(\mathscr{P}) = 0.691, \quad \lambda_{\max}(\mathscr{P}) = 1.808.$$

In addition, we have

$$||\mathrm{F}(\mathrm{t}, z)||^2 = \mathrm{F}_1^2(\mathrm{t}, z) + \mathrm{F}_2^2(\mathrm{t}, z).$$

By using $(e+h)^c \leq 2^{c-1}(e^c+h^c)$, for all $e,\ h \geq 0,\ c \geq 1$, we can compute

$$\begin{aligned} ||\mathrm{F}(\mathrm{t}, z)||^2 &\leq \frac{1}{64}(z_1^2+z_2^2) + \frac{1}{2}(e^{-2t} + \sin^2(2\pi t)), \\ &= \frac{1}{64}||z||^2 + \frac{1}{2}(e^{-2t} + \sin^2(2\pi t)). \end{aligned}$$

Hence, setting $\lambda_{\max}(\mathscr{P}) = 1.808$, $\lambda_{\min}(\mathscr{Q}) = 1$, $\Gamma = \frac{1}{64}$, and $\Psi(\mathrm{t}) = \frac{1}{2}\left(e^{-2t} + \sin^2(2\pi t)\right)$. It is simple to check whether the perturbed stochastic system (10) is globally practically uniformly exponentially stable in mean square, as shown in Figure 1 and Figure 2.

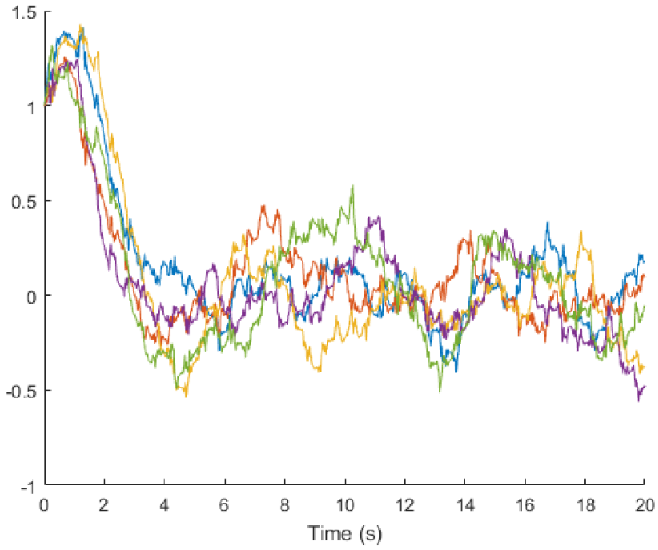

Figure 1. *Time evolution of the state $z_1(\mathrm{t})$ of the stochastic perturbed system 10, with five different Brownian motions.*

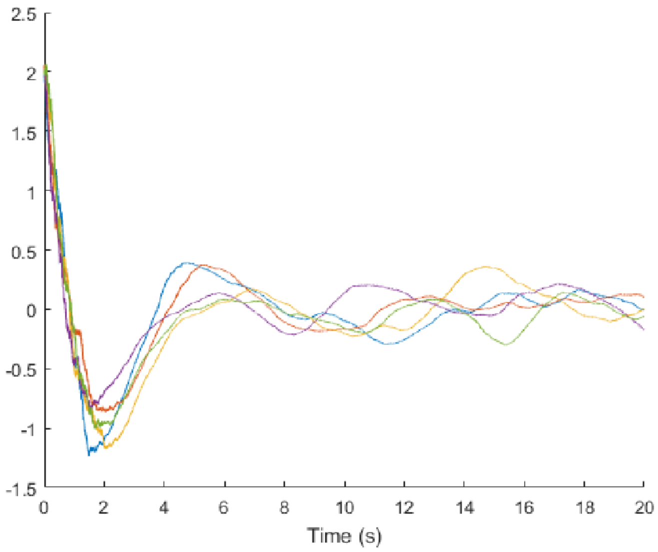

Figure 2. *Time evolution of the state $z_2(\mathrm{t})$ of the stochastic perturbed system 10, with five different Brownian motions.*

5. Conclusion

In this paper, we have studied the practical exponential stability in mean square and stabilization problem for linear time–invariant stochastic perturbed systems. We give sufficient conditions ensuring the practical uniform exponential stability of linear time–invariant stochastic perturbed systems. At the same time, we also provide general condition that guarantee the stabilization of certain classes of stochastic perturbed systems. Furthermore, a numerical example is provided to illustrate the effectiveness and advantages of our results.

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