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Fixed Point Theorems for Multivalued Contractive Mappings in Fuzzy Metric Spaces

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Abstract: In this paper, we introduce multivalued contractive mappings of Feng-Liu type in complete fuzzy metric spaces. We prove fixed point theorems for such mappings in the context of fuzzy metric spaces. We provide with an example to show that our results are more general than previously obtained results in the literature.

Keywords: Multivalued Mapping, Upper and Lower Semicontinuous, *t-norm*, Fuzzy Metric Space

1. Introduction

Due to its applications in mathematics and other related disciplines, Banach contraction principle has been generalized in many directions (for details one can see [1, 3, 4, 5, 8, 14, 19, 24, 25].

The concept of fuzzy sets was initiated by Zadeh [26] in 1965. Fuzzy metric spaces were introduced by Kramosil and Michalek [15]. Romaguera [21] introduced Hausdorff fuzzy metric on a set of nonempty closed and bounded subsets of a given fuzzy metric space. Fixed point theory in fuzzy metric spaces has been studied by a number of authors. For a wide survey we refer to ([6, 7, 11, 12, 16, 17, 18, 20, 23]) and the references therein.

Kiany et al. [14] proved fixed point theorems for multivalued fuzzy contraction maps in fuzzy metric spaces and obtained generalization of Banach contraction theorem in fuzzy metric spaces.

The aim of this paper is to obtain fixed point theorems for multivalued mapping in fuzzy metric spaces. As a results we extended the results given in ([8, 14] and reference therein) in fuzzy metric spaces.

Definition 1.1 [22] A binary operation $*:[0,1]^2 \to 0,1]$ is called a *continuous* $t-norm$ if for all $a,b,c,d \in$ [0,1];

I. $*$ is associative and commutative;
II. $*$ is continuous;
III. $a*1=a$
IV. $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$;

Definition 1.2 (compare [15]) A fuzzy metric space is a triple $(X,M,*)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0,+\infty)$ such that for all $x,y,z \in X, s,t \in (0,\infty)$:

a. $M(x,y,0)=0$;
b. $x=y$ if and only if $M(x,y,t)=1$ for all $t>0$;
c. $M(x,y,t)=M(y,x,t)$;
d. $M(x,z,t+s) \geq M(x,y,t)*M(y,z,s)$ for all $t,s \geq 0$;
e. $M(x,y,\cdot):[0,+\infty) \to [0,1]$ is left continuous.

The pair $(M,*)$ (or simply, M if no confusion arises) is said to be a fuzzy metric on X. It is well known and easy to see that for each $x,y \in X$, $M(x,y,\cdot)$ is a non-decreasing function on $[0,+\infty)$. Each fuzzy metric $(M,*)$ on a set X induces a topology τ_M on X which has a base the family of open balls

$$\{B_M(x,\varepsilon,t): x \in X,\ \ \varepsilon \in (0,1),\ \ t>0\},$$

where

$$B_M(x,\varepsilon,t)=\{y \in X: M(x,y,t)>1-\varepsilon\}.$$

Observe that a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ (with respect to τ_M) if and only if $\lim_{n\to\infty} M(x,x_n,t)=1$ for all $t>0$.

It is also well known [13] that every fuzzy metric space $(X,M,*)$ is metrizable, i.e., there exists a metric d on X whose induced topology agrees with τ_M. Conversely, if (X,d) is a metric space and we defin $M_d : X\times X\times[0,+\infty)\to[0,1]$ by

$$M(x,y,0)=0 \text{ and } M_d(x,y,t)=\frac{t}{t+d(x,y)}$$

for all $t>0$, then $(X,M_d,\wedge)$ (where $\wedge$ is minimum $t-$ norm) is a fuzzy metric space and $(M_d,\wedge)$ is called the standard fuzzy metric of (X,d) [9]. Moreover, the topology τ_{M_d} agrees with the topology induced by d.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy metric space $(X,M,*)$ is said to be a Cauchy sequence if for each $\varepsilon\in(0,1)$ there exists $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for all $n,m\geq n_0$. A fuzzy metric space $(X,M,*)$ is said to be complete ([10]) if every Cauchy sequence converges. A subset $A\subseteq X$ is said to be closed if for each convergent sequence $x_n\in A$ and $x_n\to x$, implies $x\in A$. A subset $A\subseteq X$ is said to be compact if each sequence in A has a convergent subsequence. The set of all compact subsets of X will be denoted by $K(X)$.

Lemma 1.3 [11] For all $x,y\in X$, $M(x,y,.)$ is nondecreasing.

Definition 1.4 Let $(M,X,*)$ be a fuzzy metric space, M is said to be continuous on $X^2\times[0,\infty)$ if

$$\lim_{n\to\infty} M(x_n,y_n,t_n)=M(x,y,t),$$

whenever $\{(x_n,y_n,t_n)\}$ is a sequence in $X^2\times[0,\infty)$ which converges to a point $(x,y,t)\in X^2\times[0,\infty)$; that is,

$$\lim_{n\to\infty} M(x_n,x,t)=\lim_{n\to\infty} M(y_n,y,t)=1,$$
$$\lim_{n\to\infty} M(x,y,t_n)=M(x,y,t).$$

Lemma 1.5 [11] M is a continuous function on $X^2\times[0,\infty)$.

Kiany et al. [14] introduced the following Lemma in fuzzy metric spaces.

Lemma 1.6 ([14]) Let $(X,M,*)$ be a fuzzy metric space satisfying

$$\lim_{n\to\infty} *_{i=n}^{\infty} M(x,y,th^i)=1 \tag{1}$$

for every $x,y\in X$, $t>0$ and $h>1$. Suppose $\{x_n\}$ is sequences in X satisfying

$$M(x_n,x_{n+1},\alpha t)\geq M(x_{n-1},x_n,t)$$

for all $n\in\mathbb{N}$ and $0<\alpha<1$. Then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.7 [21] Let $(X,M,*)$ be a fuzzy metric space. Then, for each $a\in X$, $B\in K(X)$ and $t>0$, there is $b_0\in B$ such that $M(a,B,t)=M(a,b_0,t)$.

Consistent with [21], we recall the notion of Hausdorff fuzzy metric induced by a fuzzy metric M as follows: For $x\in X$, and $A,B\in K(X)$ define:

$$H_M(A,B,t)=\min\{\inf_{a\in A} M(a,B,t),\inf_{b\in B} M(A,b,t)\}$$

for all $t>0$, where $M(x,A,t)=\sup_{a\in A} M(x,a,t)$. Then H_M is called the Hausdorff fuzzy metric induced by the fuzzy metric M. The triplet $(K(X),H_M,*)$ is called Hausdorff fuzzy metric space.

Definition 1.8 A function $f:X\to\mathbb{R}$ is called lower semicontinuous, if for any $\{x_n\}\subset X$ and $x\in X$, $x_n\to x$ implies $f(x)\leq\liminf_{n\to\infty} f(x_n)$. A function $f:X\to\mathbb{R}$ is called upper semicontinuous, if for any $\{x_n\}\subset X$ and $x\in X$, $x_n\to x$ implies $f(x)\geq\limsup_{n\to\infty} f(x_n)$. A multivalued mapping $T:X\to 2^X$ (collection of all nonempty subsets of X) is called upper semicontinuous, if for any $x\in X$ and a neighborhood V of $T(x)$, there is a neighborhood U of x such that for any $y\in U$, we have $T(y)\subset V$. A multivalued mapping $T:X\to 2^X$ (collection of all nonempty subsets of X) is called lower semicontinuous, if for any $x\in X$ and a neighborhood V, $V\cap T(x)\neq\varphi$, there is a neighborhood U of x such that for any $y\in U$, we have $T(y)\cap V\neq\varphi$.

2. Fixed Point Theorems for Multivalued Contractive Mappings in Fuzzy Metric Spaces

In the following theorem we obtain fixed point for multivalued mapping satisfying a contractive condition.

Let $T:X\to 2^X$ be a multivalued mapping. Define $f(x)=M(x,Tx,t)$ for $t>0$. For a positive constant $b\in(0,1)$, we define a set

$$I_b^x=\{y\in Tx\,|\,M(x,y,t)\geq M(x,Tx,bt)\}. \tag{2}$$

Theorem 2.1 Let $(X,M,*)$ be a complete fuzzy metric space and $T:X\to K(X)$ be a multivalued mapping. If there exist a constant $c\in(0,1)$ such that for any $x\in X$ there is $y\in I_b^x$ satisfying

$$M(y,Ty,ct)\geq M(x,y,t) \tag{3}$$

for $t>0$. Assume that $(X,M,*)$ satisfies (1) for some x_0 in X, then T has a fixed point provided $c<b$ and f is upper semicontinuous.

Proof: Since $T(x)\in K(X)$, by Lemma 1.7 I_b^x is nonempty for any $x\in X$ and $b\in(0,1)$. Let $x_0\in X$ be arbitrary, there exists $x_1\in I_b^{x_0}$ satisfying

$$M(x_1,Tx_1,ct)\geq M(x_0,x_1,t)$$

and for $x_1\in X$, there exists $x_2\in I_b^{x_1}$ satisfying

$$M(x_2,Tx_2,ct)\geq M(x_1,x_2,t).$$

Continuing this process, we obtain a sequence $\{x_n\}_{n\geq 0}$ in X such that $x_{n+1}\in I_b^{x_n}$ satisfying

$$M(x_{n+1},Tx_{n+1},ct)\geq M(x_n,x_{n+1},t). \tag{4}$$

On the other hand $x_{n+1}\in I_b^{x_n}$ gives

$$M(x_n,x_{n+1},t)\geq M(x_n,Tx_n,bt). \tag{5}$$

From (4) and (5) we obtain

$$M(x_{n+1},Tx_{n+1},ct)\geq M(x_n,Tx_n,bt).$$

That is

$$M(x_{n+1},Tx_{n+1},t)\geq M(x_n,Tx_n,\frac{b}{c}t). \tag{6}$$

Let $k=\frac{c}{b}$, then from (6)

$$M(x_n,x_{n+1},t)\geq M(x_{n-1},x_n,\frac{1}{k}t)\geq M(x_{n-2},x_{n-1},\frac{1}{k^2}t)\geq\ldots\geq M(x_0,x_1,\frac{t}{k^n}) \tag{7}$$

for all $n\in\mathbb{N}$ and $0<k<1$. Pick the constant $h>1$, such that $kh<1$, then $\frac{1}{h}<1$ and $\sum_{i=0}^{\infty}\frac{1}{h^i}<1$, that is $\sum_{i=n}^{m-1}\frac{1}{h^i}<1$, we get

$$t\left(\frac{1}{h^n}+\frac{1}{h^{n+1}}+\ldots+\frac{1}{h^{m-2}}+\frac{1}{h^{m-1}}\right)<t \tag{8}$$

for all $m>n$, and $h>1$. Also, we have

$$M(x_n,x_m,t)\geq M\left(x_n,x_m,t\left(\frac{1}{h^n}+\frac{1}{h^{n+1}}+\ldots+\frac{1}{h^{m-2}}+\frac{1}{h^{m-1}}\right)\right)\geq M\left(x_n,x_{n+1},\frac{t}{h^n}\right)*M\left(x_{n+1},x_{n+2},\frac{t}{h^{n+1}}\right)*\ldots*M\left(x_{m-1},x_m,\frac{t}{h^{m-1}}\right)$$

$$\geq\begin{bmatrix}M\left(x_0,x_1,\frac{t}{(kh)^n}\right)]*M\left(x_0,x_1,\frac{t}{(kh)^{n+1}}\right)*\\ \ldots*M\left(x_0,x_1,\frac{t}{(kh)^{m-1}}\right)\end{bmatrix}\geq *_{i=n}^{\infty}\left[M\left(x_0,x_1,\frac{t}{(kh)^i}\right)\right]. \tag{9}$$

Then by Lemma 6, we have

$$\lim_{m,n\to\infty}M(x_n,x_m,t)=1.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exist $x\in X$ such that $\lim_{n\to\infty}x_n=x$. From (4) and (5), it is clear that $f(x_n)=M(x_n,Tx_n,t)$ is increasing and hence converges to 1. Since f is upper semicontinuous, so we have

$$1=\limsup_{n\to\infty}f(x_n)\leq f(x)\leq 1.$$

This implies that $f(x)=1$, so $M(x,Tx,t)=1$. Hence by Lemma 7, we have $x\in Tx$.

Kiany et al. [14] gave the following corollary.

Corollary 2.2 [14] Let $(X,M,*)$ be a complete fuzzy metric. Suppose $T:X\to K(X)$ be a multivalued mapping such that

$$H_M(Tx,Ty,ct)\geq M(x,y,t) \tag{10}$$

for each $x,y\in X$, $0<c<1$ and $t>0$. Furthermore, assume that $(X,M,*)$ satisfies (1) for some x_0 and $x_1\in T(x_0)$. Then T has a fixed point.

Remark 2.3 Theorem 2.1 is a generalization of above corollary. Let T satisfies the conditions of above corollary and if f is upper semicontinuous, then from (10) for any $x\in X$, $y\in T(x)$, we get

$$M(y,T(y),ct)\geq H_M(Tx,Ty,ct)\geq M(x,y,t).$$

Hence T satisfies all the conditions of Theorem 9, the existence of fixed point has been proved. Following example shows that Theorem 2.1 is an extension of Corollary 2.2.

Example 2.4 Let $X=\left\{\frac{1}{2},\frac{1}{4},\ldots,\frac{1}{2^n},\ldots\right\}\cup\{0,1\}$,

$$M(x,y,t)=\frac{t}{t+d(x,y)}$$

and

$$d(x,y)=|x-y|,$$

for $x,y\in X$. Then $(X,M,*)$ is a complete metric space where $*:[0,1]^2\to 0,1]$ defined by $a*b=ab.$ Let $T:X\to K(X)$ be defined as given in [8]

$$T(x)=\begin{cases}\left\{\frac{1}{2^n},1\right\}, & x=\frac{1}{2^n},\text{for} n=0,1,2,...,\\ \left\{0,\frac{1}{2}\right\}, & x=0\end{cases}.$$

Since

$$\lim_{n\to\infty} *_{i=n}^{\infty} M(x,y,th^i)=M(\frac{1}{2^n},0,th^i)=\lim_{n\to\infty} *_{i=n}^{\infty}\frac{th^i}{th^i+\frac{1}{2^n}}=1$$

which implies that T satisfies (1)· Moreover

$$H_M\left(T\left(\frac{1}{2^n}\right),T0,ct\right)=\frac{ct}{ct+H\left(T\left(\frac{1}{2^n}\right),T0\right)}=\frac{ct}{ct+\frac{1}{2}}$$

and

$$M\left(\frac{1}{2^n},0,t\right)=\frac{t}{t+d\left(\frac{1}{2^n},0\right)}=\frac{t}{t+\frac{1}{2^n}}.$$

There does not exist any $0<c<1$ such that (10) is satisfied. If it exists then

$$\frac{t}{t+\frac{1}{2^n}}\le\frac{ct}{ct+\frac{1}{2}}$$

implies $c\ge 2^{n-1}$, a contradiction. On the other hand

$$f(x)=M(x,T(x),t)=\frac{t}{t+d(x,T(x))}=\begin{cases}\frac{1}{2^{n+1}}\text{if} x=\frac{1}{2^n}\\ 0\text{ if} x=0\end{cases}$$

is continuous. There exist $y\in I_{0.7}^x$ for any x such that

$$d(y,T(y))=0.5d(x,y)\le 0.6d(x,y),$$

that is $\frac{1}{0.6}d(y,T(y))\le d(x,y).$

So there exist $c=0.6<0.7$ such that

$$M(y,T(y),0.6t)=\frac{0.6t}{0.6t+d(y,T(y))}=\frac{t}{t+\frac{1}{0.6}d(y,T(y))}$$

$$\ge\frac{t}{t+d(x,y)}=M(x,y,t).$$

Then the existence of fixed point follows from Theorem 2.1.

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