

Connectedness Generalizations Using the Concept of Adherence Dominators

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To cite this article:

James Edward Joseph, Bhamini M. P. Nayar. Connectedness Generalizations Using the Concept of Adherence Dominators. *American Journal of Applied Mathematics*. Vol. 10, No. 2, 2022, pp. 43-50. doi: 10.11648/j.ajam.20221002.13

Received: March 6, 2022; **Accepted:** March 30, 2022; **Published:** April 23, 2022

Abstract: An *adherence dominator* on a topological space X is a function π from the collection of filterbases on X to the collection of closed subsets of X satisfying $\mathcal{A}\Omega \subset \pi\Omega$ where $\mathcal{A}\Omega$ is the adherence of Ω and $\pi\Omega = \bigcap_{\Omega} \pi F = \bigcap_{\sum F, F \in \Omega} \pi V$, where $\sum F$ represents the collection of open sets containing F . The π -adherence may be adherence, θ -adherence, u -adherence s -adherence, f -adherence, δ -adherence, etc., of a filterbase. Pervin defined a partition (or a separation) of a set in a topological space as a pair of subsets (P, Q) satisfying $P \cap clQ = clP \cap Q = \emptyset$, where clP represents the closure of P and a set K is said to be connected if $K = \emptyset$ or $K \neq P \cup Q$ where (P, Q) is a partition. In this paper, a π partition (or a π separation) is a pair of subsets (P, Q) satisfying $P \cap \pi Q = \pi P \cap Q = \emptyset$ where π is an adherence dominator and a subset K of a space X is π connected relative to X if $K = \emptyset$ or there is no π partition (P, Q) such that $K = P \cup Q$. This paper investigates these new forms of connectedness. Theorems due to A. D. Wallace and J. D. Kline are generalized. Generalizations of C -compact spaces and functionally compact spaces are also presented.

Keywords: Filters, Adherence Dominators, Connectedness, π separation, π closed

1. Introduction

An *adherence dominator on a topological space* is a function π from the set of filterbases to the set of closed subsets satisfying $\mathcal{A}\Omega \subset \pi\Omega$, where $\mathcal{A}\Omega$ is the adherence of Ω and $\pi\Omega = \bigcap_{\Omega} \pi F = \bigcap_{\sum F, F \in \Omega} \pi V$, where $\sum F$ represents the collection of open sets containing the set F . If π is an adherence dominator, a nonempty subset B in a space X is π closed if $\pi B = B$ and a space to be a π space if $\{x\}$ is π closed for every $x \in X$. A partition is defined in a topological space as a pair of subsets (P, Q) satisfying $P \cap clQ = clP \cap Q = \emptyset$ where clP represents the closure of P and K is said to be connected if $K = \emptyset$ or $K \neq P \cup Q$ where (P, Q) is a partition [19]. For a space X and $P \subseteq X$, if $P \neq \emptyset$, $clP \subseteq \pi P$.

The π -adherence may be adherence, θ -adherence, u -adherence s -adherence, f -adherence δ -adherence, etc., of a filterbase [22, 6, 7, 10, 20, 9, 11, 8, 12, 21]. These closure operators are not individually stated here. Please refer to the appropriate references for the definitions of each of these

closure operators.

This article is divided into six sections. In this section for introduction, concepts which motivated the study of connectedness via an adherence dominator are presented; next section on π connectedness, π connected sets and their properties are given. In the third section, generalizations of continuity are presented. The fourth section gives some generalizations of compactness through adherence dominator and in the fifth section components relative to a space are presented through adherence dominator. Sixth section remarks on conclusion of this study.

Following result is stated without proof as it can be easily verified.

Theorem 1 A T_1 space X is a π space if and only if $\pi \sum \{x\} = \{x\}$ for every $x \in X$, where $\sum \{x\}$ represents the collection of open sets containing x and $\pi \sum \{x\} = \bigcap_{\sum \{x\}} \pi V$.

2. π Connectedness

The concept of a π connected subset is defined in the model of the connected subset, using π closure, instead of closure of a set.

Definition 1 A subset K is called π connected relative to X if $K = \emptyset$ or $K \neq P \cup Q$, $P \cap \pi Q = \pi P \cap Q = \emptyset$, where π is an adherence dominator on X , that is to say $K = \emptyset$ or K has no π partition (or π separation).

It is clear that connected subsets are π connected relative to the space. The following two examples illustrate the following: They are given here to show that (1) π connected relative to a space need not be connected and (2) that these two classes may coincide in spaces which are not regular, using $\pi\Omega = ad_\theta\Omega$, where $ad_\theta\Omega$ represents the θ -adherence of Ω .

Example 1 [2]. Let I be the unit interval $[0, 1]$, $Y = I \times \{0\}$, $X = I \times I$ with the topology generated by the following base for the open sets: (1) the relative open sets from the plane in $X - Y$ (2) and for $x \in Y$, sets of the form $(V \cap (X - Y)) \cup \{x\}$ where V is open in the plane with $x \in V$. It is clear that Y is discrete in the relative topology from X and hence is not connected. Suppose $\pi = cl_\theta$, the θ -closure operator, and that (P, Q) a π separation relative to X , and $Y = P \cup Q$. Choose $(r, 0) \in P$ and without loss of generality, assume that there is an $s \in I$ with $r < s$ and $(s, 0) \in Q$. Let

$$c = \sup\{r \in I : r < s \text{ and } (r, 0) \in P\}.$$

We see easily that $(c, 0) \in \pi P$. Hence $(c, 0) \in P$. Since it is readily seen that $(c, 0) \in \pi Q$, a contradiction is reached and Y is π connected relative to X ($\pi = cl_\theta$).

Example 2 [2]. Let E be the set of even positive integers, O the set of odd positive integers, and let $X = \{0\} \cup E \cup O \cup \{j + n^{-1} : j, n \in (E \cup O) - \{1\}\}$ be endowed with the topology from the following open set base: (1) the relative open sets from the reals in $X - \{0, 1\}$, (2) all subsets of the form $\{0\} \cup \{j + n^{-1} : n \in E, j \geq j_0\}$ where $j_0 > 1$ and (3) all subset of the form $\{1\} \cup \{j + n^{-1} : n \in O - \{1\}, j \geq j_0\}$ where $j_0 > 1$. It is not difficult to see that the collection of π connected subsets and the collection of connected subsets are both the collection of singletons ($\pi = cl_\theta$).

The adherence operator π subsumes the adherence of a filterbase based on different closure operators. As we have seen above, a π connected relative to the space need not be connected. However, note that if for a space X , the collection of closed sets coincide with the collection of π closed sets, then π connected sets will be connected. Therefore, investigating on additional condition(s) which will make sets which are π connected relative to the space to be connected, one notice that neither θ -closure operator nor u -closure operator are Kuratowski closure operators [13]. In the cited paper, it was shown that if the topology on the space agree with the semiregularization topology T_s , that is, the topology generated by the regularly open subsets of the space, where the collection of closed sets is the collection of regularly closed sets, then $cl_\theta(cl_\theta A) = cl_\theta A$ and $cl_u(cl_u A) = cl_u A$. (A set is regularly

open (regularly closed), if it is the interior of a closed (closure of an open) set. Complement of a regularly open set is regularly closed. Also, for any space X , if T_k denotes the topology on X generated by the θ -closures of regularly closed subsets and T_u denotes the topology generated by u -closed subsets, then $T_k = T_u$ [13]. So, when π -closure is a Kuratowski operator for a space X , if T_π denotes the topology generated by the π -closed subsets of X , then for (X, T_π) , the class of π connected relative to X subsets will coincide with the class of connected subsets of X . Note that the space in Example 2 is semiregular and as is observed, in that the collection of connected sets coincide with the collection of π connected sets, where $\pi = cl_\theta$.

The following concept and results are used in the sequel and hence are included in this section

Definition 2 A set $A \subseteq X$ is said to be π rigid, if each filter base Ω on X satisfying the property that $F \cap \pi V \neq \emptyset$, for all $F \in \Omega$ and $V \in \Sigma A$ also satisfies that $\pi\Omega \cap A \neq \emptyset$ [4].

The above definition of a π rigid set subsumes the concepts of θ -rigid, u rigid, s rigid sets when the adherence dominator π is the θ -closure or u -closure or s -closure operator respectively [4].

It is to be noted that in a π space, $\pi\{x\} = \{x\}$ for each $x \in X$ and for any set $A \subseteq X$, $\bigcup_A \pi\{x\} \subseteq \pi A$. It can be shown that $\pi A = \bigcup_A \pi\{x\}$ for any π rigid $A \subseteq X$.

Theorem 2 If A is π rigid, then $\pi A = \bigcup_A \pi\{x\}$.

Proof Let A be π rigid and let $x \in \pi A$. Then the constant net $\{x\}$ is frequently in πV for all $V \in \Sigma A$. Hence there is a $y \in A$ such that $\{x\}$ is frequently in πV for all $V \in \Sigma\{y\}$. So, $x \in \pi\{y\}$. Therefore, $\pi A \subseteq \bigcup_A \pi\{x\}$ and hence $\pi A = \bigcup_A \pi\{x\}$.

The following cahacterization of π rigid subsets is used later in this article.

Theorem 3 A subset K of a space X is π rigid if and only if for every cover Ω of K with open subsets of X there is a finite $\Omega^* \subset \Omega$, $K \subset \text{int}(\bigcup_{\Omega^*} \pi V)$.

Proof Suppose that A is π rigid and Ω is an open cover of A with subsets of X . Suppose that for any finite subfamily Ω^* , of Ω , $A \not\subseteq \text{int}(\bigcup_{\Omega^*} \pi F)$. Then $A \cap (X - \text{int}(\pi F)) = A \cap cl(X - \pi F) \neq \emptyset$, for each $F \in \Omega$. The family $\mathcal{F} = \{A \cap cl(X - \pi F), F \in \Omega\}$ is a filterbase on X and $(X - \pi F) \cap \pi V \neq \emptyset$ for $V \in \Sigma(A)$. Hence $\pi\mathcal{F} \cap A \neq \emptyset$. So, $(\cap(X - F)) \cap A = (X - \bigcup_{\Omega} F) \cap A \neq \emptyset$, a contradiction.

Conversely, suppose that the open cover condition holds and Ω is a filterbase on X with $\pi\Omega \cap A = \emptyset$. Then there is an $F \in \Omega$ such that $\pi F \cap A = \emptyset$ which implies that $\pi V \cap F = \emptyset$ for some $V \in \Sigma(A)$, $F \in \Omega$. Hence A is π rigid.

Definition 3. Let (X, τ) be a topological space and let $\tau \subset \tau^*$. Then τ^* will be called a *simple extension* of τ if there exists an $A \notin \tau$ such that $\tau^* = \{U \cup (V \cap A), \text{ with } U, V \in \tau\}$ and $\tau^* = \tau(A)$, a simple extension of τ through A [18].

Theorem 4 The following statements are equivalent for a T_1 space X :

1. The space X is a π space;
2. Each π rigid subset of X is π closed;
3. Each compact subset of X is π closed;

4. Each continuous function on X to a π space maps compact sets onto π closed sets;
5. Each continuous bijection onto X maps compact subsets onto π closed sets.

Proof (1) \Rightarrow (2). Let A be π rigid. Then $\pi A = \bigcup_A \pi(\{x\}) = \bigcup_A \{x\} = A$, since X is a π space. So, A is π closed.

(2) \Rightarrow (3). A compact subset is π rigid.

(3) \Rightarrow (4) \Rightarrow (5). Obvious.

(5) \Rightarrow (1). Let $x \in X$ and let X^* be X with the simple extension of X through the set $\{x\}$ [18]. The identity function $g : X^* \rightarrow X$ is a continuous bijection onto X and $\{x\}$ is compact in X^* . So $\pi g(x) = \pi\{x\} = \{x\}$.

Corollary 1 The following statements are equivalent for a space X :

1. The space X is a Hausdorff (Urysohn) [regular] space;
2. Each θ -rigid ($urigid$) [$srigid$] subset of X is θ closed ($uclosed$) [$sclosed$];
3. Each compact subset of X is θ closed ($uclosed$) [$sclosed$];
4. Each continuous function on X to a Hausdorff (Urysohn) [regular] space maps compact sets onto θ closed ($uclosed$) [$sclosed$] sets;
5. Each continuous bijection onto X maps compact subsets onto θ closed ($uclosed$) [$sclosed$] sets.

The proof of the following result is easy and follows in the same line of proof as for a connected set A .

Theorem 5 The following statements are equivalent for $A \subset X$:

- (1) A is π connected relative to X ;
- (2) For each two points $x, y \in A$, there is a π connected relative to X set $B \subset A$ with x, y in B ;
- (3) If (P, Q) is a π separation relative to X and $A \subset P \cup Q$ then either $A \subset P$ or $A \subset Q$.

Theorem 6 The following statements hold for spaces X, Y :

- (1) Connected subsets of a space X are π connected relative to X ;
- (2) A pair V, W of disjoint open subsets of X is a π separation relative to X ;
- (3) If A is π connected relative to X and $A \subset B \subset \pi A$, then B is π connected relative to X ;
- (4) If Ω is a family of subsets π connected relative to X no pair of which is a π separation relative to X then $\bigcup_{\Omega} F$ is π connected;
- (5) If Ω is a family of subsets π connected relative to X and there is an $F_0 \neq \emptyset, F_0 \in \Omega, (F_0, F)$ fails to be a π separation relative to X for each $F \in \Omega$, then $\bigcup_{\Omega} F$ is π connected. relative to X ;
- (6) If F_n is a sequence of π connected relative to X subsets and (F_n, F_{n+1}) fails to be a π separation relative to X for all n then $\bigcup_{n \in \mathbb{N}} F_n$ is π connected relative to X ;
- (7) If A, B are π connected relative to X and Y respectively, then $A \times B$ is π connected relative to $X \times Y$;
- (8) If A is π connected relative to X and $X_0 \subset X$ satisfies $A \cap \text{int} X_0 \neq \emptyset, A \cap \text{int}(X - X_0) \neq \emptyset$ then $A \cap \text{bd} X_0 \neq \emptyset$;
- (9) If Ω is a family of π connected relative to X subsets and

$\bigcap_{\Omega} F \neq \emptyset$ then $\bigcup_{\Omega} F$ is π connected.

Proof Only proofs of (3) and (7) are given here.

Proof of (3). Let (P, Q) be a π separation relative to X and suppose $B \subset P \cup Q$. Therefore $A \subset P$ or $A \subset Q$. So $B \subset \pi P$ or $B \subset \pi Q$.

Proof of (7): Let $(x, y), (u, v) \in A \times B$. Then $(x, v) \in (A \times \{v\}) \cup (\{x\} \times B)$ where both $(A \times \{v\})$ and $(\{x\} \times B)$ are π connected relative to $X \times Y$ since π connectivity is a topological invariant, as is shown in Theorem 11. It follows that

$$(x, v) \in (A \times \{v\}) \cap \{x\} \times B$$

and

$$(x, y), (u, v) \in (A \times \{v\}) \cap (\{x\} \times B) \subset A \times B.$$

Hence, in view of Theorem 5 (2), the proof is complete.

Theorem 7 If $\{X(n)\}_{\Lambda}$ is a family of spaces and $\{A(n)\}_{\Lambda}$ is a family of sets such that $A(n)$ is π connected relative to $X(n)$ for each $n \in \Lambda$ then $\prod_{\Lambda} A(n)$ is π connected relative to $\prod_{\Lambda} X(n)$

Proof Let (P, Q) be a π separation relative to $\prod_{\Lambda} X(n)$ such that

$$\prod_{\Lambda} A(n) \subset P \cup Q.$$

Let $x \in \prod_{\Lambda} A(n)$, and for each $n \in \Lambda$, Let $P_n : \prod_{\Lambda} X(n) \rightarrow X(n) \cap P_n(P)$ be the projection intersected with the projection of P . There is a basic open set $V = \bigcap_{\Lambda(x)} p_n^{-1} W(n)$ with $x \in V, \Lambda(x)$ a finite subset of Λ , and $Q \cap \pi V = \emptyset$. Choose $y \in \prod_{\Lambda} A(n)$ and let

$$J(y) = \{z \in \prod_{\Lambda} A(n) : z(n) = y(n), n \in \Lambda - \Lambda\{x\}\}.$$

$J(y)$ is homeomorphic to $\prod_{\Lambda(x)} A(n)$ and Theorem 6(7) may be used with induction to show that $\prod_{\Lambda(x)} A(n)$ is π connected relative to $\prod_{\Lambda} X(n)$. This implies $J(y) \subset P$ since we see z defined by $z(n) = x(n)$ when $n \in \Lambda\{x\}, z(n) = y(n)$ when $n \in \Lambda - \Lambda(x)$, it follows that $z \in J(y) \cap V \subset P$.

The following theorem is a generalization of a result of J. R. Kline remarking on a paper of Knaster and Kuratowski [16], [17].

Definition 4. A set X is *totally π disconnected* if the singletons are the only non-empty π connected subsets of X .

Theorem 8 If X is π connected relative to X and for $y \in X, X - \{y\}$ is totally π disconnected then $X - \{x\}$ is π connected relative to X for every $x \neq y, x \in X$.

Proof For every $A \subset X - \{y\}$, with more than one point, there is a π separation (P, Q) of A . Let $x \in X - \{y\}$. The set $X - \{y\}$ has at least two elements x, z . If there is a π separation (P, Q) of $\{x, z\}$, for each $z \notin \{x, y\}$, we get a π separation for the π connected set X , a contradiction. Hence $X - \{x\}$ is π connected relative to X for every $x \neq y$.

3. Genralizations of Continuity Through Adherence Dominators

Below, some generalizations of continuous functions are provided in terms of adherence dominator. One of the well-known characterizations of continuity of a function $f : X \rightarrow Y$ is that $f(clA) \subseteq clf(A)$ for each subset A of X . It is this characterization which is used as a model to define the notion of a π continuous function.

Definition 5. A function $f : X \rightarrow Y$ is π continuous if $f(\pi\Omega) \subset \pi f(\Omega)$ for every filterbase Ω . A function $f : X \rightarrow Y$ is *weakly- π continuous* if $f(A\Omega) \subset \pi f(\Omega)$ for every filter Ω . A function $f : X \rightarrow Y$ is $\theta\pi$ continuous if $f(\theta\Omega) \subset \pi f(\Omega)$ for every filter Ω , where $\theta\Omega$ is used here for the θ adherence of Ω and $A\Omega$ denotes the adherence of Ω .

Theorem 9 The following are equivalent:

1. The function $f : X \rightarrow Y$ is π continuous;
2. The function $f : X \rightarrow Y$ satisfies $f(\pi A) \subset \pi f(A)$ for each nonempty $A \subset X$;
3. The function $f : X \rightarrow Y$ satisfies $\pi f^{-1}(B) \subset f^{-1}(\pi B)$ for each nonempty $B \subset Y$;
4. The function $f : X \rightarrow Y$ satisfies $\pi f^{-1}(\Omega) \subset f^{-1}(\pi\Omega)$ for each filterbase Ω on Y ;
5. The function $f : X \rightarrow Y$ satisfies $\bigcap_{\sum A} \pi f^{-1}(V) \subset f^{-1}(\pi A)$ for each $A \subset X$.

Proof (1) \Leftrightarrow (2). For (1) \Rightarrow (2), consider the filterbase $\{A\}$, for $A \subseteq X$. For (2) \Rightarrow (1), note that for each $F \in \Omega$, $f(\pi F) \subseteq \pi f(F)$ and hence (1) follows.

(2) \Leftrightarrow (3). Assume (2). Let $B \subset Y$. Then $f(\pi f^{-1}(B)) \subset \pi f(f^{-1}(B)) \subset \pi B$. So, $\pi f^{-1}(B) \subset f^{-1}(\pi B)$.

Now, suppose we have (3). let $A \subset X$. Clearly, $A \subseteq f^{-1}(f(A))$. So, $\pi A \subset \pi f^{-1}(f(A)) \subset f^{-1}(\pi f(A))$. Hence, $f(\pi A) \subset \pi f(A)$.

Proofs of (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (3) are clear.

Corollary 2 If X, Y are spaces and π is an adherence dominator on X , a function $f : X \rightarrow Y$ is π continuous if and only if $\pi f^{-1}(W) \subset f^{-1}(\pi W)$ for each open W .

Proof The function $f : X \rightarrow Y$ satisfies $\pi f^{-1}(B) \subset f^{-1}(\pi B)$ for each $B \subset Y$.

For weakly π continuous ($\theta\pi$ continuous) functions, we have the following:

Theorem 10 The following are equivalent for spaces X, Y and function $f : X \rightarrow Y$.

1. $f : X \rightarrow Y$ is weakly π continuous ($\theta\pi$ continuous);
2. $cl f^{-1}(B) \subset f^{-1}(\pi B)$ ($cl_{\theta} f^{-1}(B) \subset f^{-1}(\pi B)$) for each non-empty $B \subset Y$;
3. $f(cl(A) \subset \pi f(A))$ ($f(cl_{\theta}(A) \subset \pi f(A))$) for every $A \subset X$;
4. The function $f : X \rightarrow Y$ satisfies $\mathcal{A}f^{-1}(\Omega) \subset f^{-1}(\pi\Omega)$ ($ad_{\theta} f^{-1}(\Omega) \subset f^{-1}(\pi\Omega)$) for each filterbase Ω on Y ;
5. The function $f : X \rightarrow Y$ satisfies $\bigcap_{\sum A} cl f^{-1}(V) \subset f^{-1}(\pi A)$ ($\bigcap_{\sum A} cl_{\theta} f^{-1}(V) \subset f^{-1}(\pi A)$) for each $A \subset X$.

Proof Proof is similar to the proof of Theorem 9.

Theorem 11 If $f : X \rightarrow Y$ is a π continuous function and K is π connected relative to X , then $f(K)$ is π connected relative to Y .

Proof Let (P, Q) be a π partition such that $f(K) \subset P \cup Q$, $P \cap \pi Q = \emptyset$, $Q \cap \pi P = \emptyset$. Then $K \subset f^{-1}(P) \cup f^{-1}(Q)$. However, f being π continuous, $\pi f^{-1}(P) \cap f^{-1}(Q) \subset f^{-1}(\pi P) \cap f^{-1}(Q) = f^{-1}(Q \cap \pi P) = \emptyset$. Similarly, $\pi f^{-1}(Q) \cap f^{-1}(P) = \emptyset$. That is, $f^{-1}(P)$ and $f^{-1}(Q)$ are π separated. Thus, K being π connected, $K \subset f^{-1}(P)$ or $K \subset f^{-1}(Q)$ according to Theorem 5(3). So, $f(K)$ is π connected.

4. Some Generalizations of Compactness

Before introducing the generalizations compactness through adherence dominators, the following results of filter convergence, which will be used in the sequel, are provided. The concepts of functionally compact spaces and C-compact spaces are some of the generalizations of compactness considered here. See [1], [23].

Theorem 12 If \mathcal{F} is a filter on a π space X and $\mathcal{O} = \{V \text{ open in } X : F \subset V \text{ for some } F \in \mathcal{F}\}$, then \mathcal{O} is an open filter and $\pi\mathcal{F} = \pi\mathcal{O}$.

Proof \mathcal{O} is an open filter and $\mathcal{O} \subseteq \mathcal{F}$. So, $\pi\mathcal{F} \subset \pi\mathcal{O}$. For the reverse inclusion, if $x \notin \pi\mathcal{F}$, then $x \notin \pi F$ for some $F \in \mathcal{F}$. Therefore, there exist $F \in \mathcal{F}$ and $V \in \sum\{x\}$ such that $\pi V \cap F = \emptyset$. Therefore, $F \subset X - \pi V$. Hence, $X - \pi V \in \mathcal{O}$, and $\pi V \cap (X - \pi V) = \emptyset$. So $x \notin \pi\mathcal{O}$.

Corollary 3 If \mathcal{F} is an ultrafilter on a π space X and $\mathcal{O} = \{V \text{ open in } X : F \subset V \text{ for some } F \in \mathcal{F}\}$, then \mathcal{O} is an open filter and $\pi\mathcal{F} = \pi\mathcal{O}$. [3]

Proof Note that $\mathcal{O} \subset \mathcal{F}$.

Definition 6. If \mathcal{O} is an open filter, then $\pi\mathcal{O} = \bigcup_{\mathcal{O}} \pi\{x\}$.

Corollary 4. If \mathcal{F} is a filter on a π space X , then $\pi\mathcal{F} = \bigcup_{\mathcal{F}} \pi\{x\}$.

Proof If \mathcal{O} is an open filter, then $\pi\mathcal{O} = \bigcup_{\mathcal{O}} \pi\{x\}$. Theorem 12 then completes the proof.

Definition 7. If π is an adherence dominator on a π space X , then X is π closed if $\pi\Omega \neq \emptyset$ for each filterbase Ω on X .

The following result is immediate.

Theorem 13 If X is π closed and Ω is a filterbase on X , then $\pi\Omega$ is quasi π closed relative to X , where a set X is said to be quasi π closed if $\pi\Omega \neq \emptyset$ for each filterbase Ω on X , but X is not necessarily a π space.

Corollary 5. If X is an H-closed (Urysohn-closed)[regular-closed] space and Ω is a filterbase on X then $adh_{\theta}\Omega(adh_u\Omega)[adh_s\Omega]$ is quasi H-closed (quasi Urysohn-closed) [quasi regular-closed] relative to X . [2]

Definition 8. If π is an adherence dominator on a space X , then X is *rim π closed* if each $x \in X$ has a base of open sets with π closed boundaries.

Theorem 14 If X is a space, A is a nonempty compact subset of X and Ω is a filter on X such that $A \cap F \neq \emptyset$ for every $F \in \Omega$, there exists $x \in A$ such that $F \cap V \neq \emptyset$, $F \in \Omega$, $V \in \sum\{x\}$.

Proof Suppose that for each $x \in A$, $F_x \cap V_x = \emptyset$, $F_x \in \Omega$, $V_x \in \sum\{x\}$. Then, there exists finite $B \subset A$ such that

$A \subset \bigcup_B V_x = V, F = \bigcap_B F_x \in \Omega, V \in \sum A, F \cap V = \emptyset$. Therefore, there exists $x \in A$ such that $F \cap V \neq \emptyset, F \in \Omega, V \in \sum\{x\}$.

Theorem 15 If X is a π space, A is a nonempty compact subset of X and \mathcal{U} is an ultrafilter on X such that $A \cap F \neq \emptyset$ for every $F \in \mathcal{U}$ there exists $x \in A$ such that $\mathcal{U} \rightarrow x$.

Proof Follows easily from Theorem 14, for an ultrafilter \mathcal{U} on X .

Theorem 16 The following are equivalent for a π space X and adherence dominator π on X .

1. X is compact;
2. Each nonempty closed subset of X is π closed, and $\pi\mathcal{U}$ is compact for each ultrafilter \mathcal{U} on X ;
3. The boundary bdV is π closed for each open, non-empty proper subset of X and $\pi\mathcal{U}$ is compact for each ultrafilter \mathcal{U} on X ;
4. X is rim π closed, and $\pi\mathcal{U}$ is compact for each ultrafilter \mathcal{U} on X .

Proof (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4). Immediate.

(4) \Rightarrow (1). Let \mathcal{U} be an ultrafilter on X and let $V \in \sum \pi\{x\}, x \in \pi\mathcal{U}$. Then $F - V = \emptyset$ for some $F \in \mathcal{U}$. If not, $V, bdV \in \mathcal{U}$, a contradiction. So $\mathcal{U} \rightarrow \pi\{x\}$. Since $\pi\{x\}$ is a closed, and therefore compact subset of $\pi\mathcal{U}$, there exists $z \in \pi\{x\}, \mathcal{U} \rightarrow z$, in view of Theorems 14 and 15.

Recall that a Hausdorff space X is *C-compact* (functionally compact)[1] if each closed set is an H-set (if each open filterbase Ω on X satisfying $\mathcal{A}\Omega = I\Omega$ is an open set base for $I\Omega$) [23], [1]. Below we define and characterize these concepts using adherence dominators.

Definition 9. If X is a space, a non-empty set $A \subset X$ is a π set if each filterbase Ω on A satisfies $\pi\Omega \cap A \neq \emptyset$.

Definition 10. A space X is π C-compact if every non-empty closed set $A \subset X$ is a π set.

Definition 11. A filterbase \mathcal{F} is said to π converge to x (to a set A), if for each $V \in \Sigma x$ (ΣA), there is an $F \in \mathcal{F}$ such that $F \subseteq \pi V$.

Theorem 17 If X is a π closed space and $\emptyset \neq A \subset X$, then πA is a π set.

Proof Let Δ be a filterbase on πA . Since for any filterbase Ω on $X, \pi\Omega \cap \pi A \neq \emptyset$, and Δ is a filterbase on $X, adh\Delta \cap \pi A \neq \emptyset$.

Proposition 1. Let X be a space and $A \subset X$ be π closed. If (P, Q) is a π separation of A relative to X , then P and Q are π closed in X .

Proof Let $A = P \cup Q$, with $P \cap \pi Q = \pi P \cap Q = \emptyset$. Since A is π closed, $\pi A = \pi P \cup \pi Q = A = P \cup Q$. Since $P \cap \pi Q = \pi P \cap Q = \emptyset$, this implies that $\pi P \subseteq P$ and $\pi Q \subseteq Q$.

The following Theorem gives characterizations of π C-compact spaces in terms of ultrafilters.

Theorem 18 The following are equivalent for a π space X :

1. X is π C-compact;
2. For each closed subset $B \subset X$ each ultrafilter $\mathcal{U}, B \in \mathcal{U}, \mathcal{U}$ π converges to some point of B ;
3. $\bigcap_B \pi B$, where \mathcal{B} is the collection of closed sets in \mathcal{U} , is a singleton for each ultrafilter \mathcal{U} .

Proof (1) \Rightarrow (2). Follows from the fact that in a π C-compact

space, every non-empty closed set is a π set. Hence the ultrafilter \mathcal{U} has non-empty π adherence in B and therefore, \mathcal{U} π converges to some point of B .

(2) \Rightarrow (3). Let \mathcal{U} be an ultrafilter on X and let \mathcal{B} be the collection of closed sets in \mathcal{U} . Suppose that $\{x, y\} \subseteq \bigcap_B \pi B$, with $x \neq y$. Then, in view of (2) and since πB being closed, $\mathcal{U} \rightarrow x$ and $\mathcal{U} \rightarrow y$, a contradiction. Hence (3).

(3) \Rightarrow (1). Let A be a closed set and let Ω be a filterbase on A . Let \mathcal{U} be an ultrafilter containing Ω . In view of (3), there is a singleton set $\{x\} = \bigcap_B \pi B$, where \mathcal{B} is the collection of closed sets in \mathcal{U} . Since A is closed and Ω is a filterbase on $A, x \in A$. Hence A is a π set.

Corollary 6. The following are equivalent for a Hausdorff space X :

1. X is C-compact;
2. For each closed subset $B \subset X$ and each ultrafilter \mathcal{U} with $B \in \mathcal{U}, \mathcal{U}$ θ converges to some point of B ;
3. $\bigcap_B cl_\theta B$, where \mathcal{B} is the collection of closed sets in \mathcal{U} , is a singleton for each ultrafilter \mathcal{U} .

Theorem 19 In a π C-compact space, for any two disjoint π closed sets P and Q , there are open sets V and W such that $P \subset V, Q \subset W, \pi V \cap \pi W = \emptyset$.

Proof Let P and Q be two disjoint π closed sets of the π C-compact space X . Since each π closed set is closed, P and Q are π sets. Suppose that for each $V \in \Sigma P$ and $W \in \Sigma Q$, with $\pi V \cap \pi W \neq \emptyset$. Then $\Omega = \{\pi V \cap \pi W, V \in \Sigma P \text{ and } W \in \Sigma Q\}$ is a filterbase in πP as well as in πQ and has non-empty adherence in πP and in πQ , a contradiction since $\pi P \cap \pi Q = \emptyset$.

Corollary 7. In a C-compact space, any two disjoint θ -closed sets are separated by disjoint open subsets.

Theorem 20 Let X be a π C-compact space and let Ω be a filterbase of π connected relative to X subsets. Then $\pi\Omega$ is π connected relative to X if $\pi\Omega$ is π closed in X .

Proof Let (P, Q) be a π separation relative to X of $\pi\Omega$. Then P and Q are disjoint and π closed in X from the Proposition 1. Since X is π C-compact, there are sets $V \in \Sigma P, W \in \Sigma Q, V \cap \pi W = \pi V \cap W = \emptyset$. Note that $\pi\Omega$ is π closed and hence is closed in the C-compact space X . Also $\pi\Omega \subseteq V \cup W$. So there is an $F \in \Omega, F \subset V \cup W$. Since F is π connected relative to X and (V, W) is a π separation relative to $X, F \subset V$ or $F \subset W$. Hence $\pi F \subset \pi V$ or $\pi F \subset \pi W$. So $\pi\Omega \subset \pi V$ or $\pi\Omega \subset \pi W$. Therefore, $\pi\Omega \cap P = \emptyset$ or $\pi\Omega \cap Q = \emptyset$, a contradiction.

Corollary 8. Let X be a C-compact space and let Ω be a filterbase of θ connected relative to X subsets. Then $ad_\theta \Omega$ is θ connected relative to X if $ad_\theta \Omega$ is θ closed in X . [2]

Arguments similar those in the proof of the last theorem lead to the following.

Theorem 21 Let X be a π C-compact space and let Ω be a filterbase of connected subsets. Then $\pi\Omega$ is connected if open and closed subsets are π closed in X .

Definition 12. A π space X is π functionally compact if each open filterbase Ω of the space satisfying the condition $\pi\Omega = I\Omega$ is an open set base for $I\Omega$, where $I\Omega$ represents the intersection of members of Ω .

Theorem 22 The following are equivalent for a π space X :

1. The space X is π functionally compact;
2. Each open filterbase Ω on X satisfying $\pi\Omega = I\Omega$ converges to Ω ;
3. Each filterbase Ω on X satisfying $\pi\Omega = I\Omega$ converges to $I\Omega$.

Proof (1) \Rightarrow (2). Follows from the definition of π functionally compact spaces. (2) \Rightarrow (3). Note that $\pi\Omega = \mathcal{A}_\Omega(\Sigma\pi F)$.

(3) \Rightarrow (1). Follows easily.

Corollary 9. A Hausdorff space X is functionally compact if and only if each filterbase Ω on X with $ad_\theta\Omega = I\Omega$ converges to $I\Omega$. (Theorem 3.12 [13])

When the π adherence represents u adherence, we have the following result.

Corollary 10 [13] The following are equivalent for a Urysohn space X :

1. X is u functionally compact;
2. Each open filterbase Ω on X with $ad_u\Omega = I\Omega$ converges to $I\Omega$;
3. Each filterbase Ω on X satisfying $ad_u\Omega = I\Omega$ converges to $I\Omega$.

The Wallace Theorem states that if A and B are compact sets in X and Y respectively and $A \times B \subset H, H$ open in $X \times Y$, there exists V open in X, W open in Y such that $A \subset V, B \subset W, V \times W \subset H$. (see Theorem 12 p. 142 [15])

The following is a restatement of the Wallace Theorem in terms of closures.

Theorem 23 Let X, Y be spaces and A, B be compact subsets of X, Y respectively. Let $K \subset X \times Y$ satisfy $(A \times B) \cap clK = \emptyset$. Then there are $V \in \Sigma A, W \in \Sigma B$ satisfying $(V \times W) \cap clK = \emptyset$.

The next theorems generalize Wallace's theorem to π rigid subsets.

Theorem 24 Let X, Y be spaces and let $B \subset Y$ be π rigid. Let $x \in X, K \subset X \times Y$ satisfy $(\{x\} \times B) \cap \pi K = \emptyset$. Then there are open sets $V \in \Sigma\{x\}, W \in \Sigma B$ such that $(V \times W) \cap \pi K = \emptyset$.

Proof For each $y \in B$, there are open sets $V(y) \in \Sigma\{x\}, W(y) \in \Sigma(y), (V(y) \times W(y)) \cap \pi K = \emptyset$. There is a finite $B^* \subset B$ with

$$B \subset \text{int} \left(\bigcup_{B^*} \pi W(y) \right) = W,$$

in view of Theorem 3. Let $V = \bigcap_{B^*} V(y)$. Then $V \in \Sigma\{x\}, W \in \Sigma B$. Therefore $(V \times W) \cap \pi K = \emptyset$.

Theorem 25 Let X, Y be spaces and A, B be π rigid subsets of X, Y respectively. Let $K \subset X \times Y$ satisfy $(A \times B) \cap \pi K = \emptyset$. Then there are $V \in \Lambda A, W \in \Lambda B$ satisfying $(V \times W) \cap \pi K = \emptyset$, where $\Lambda A = \Sigma \pi A$.

Proof From the proof of the last theorem, Theorem 24, for each $x \in X$ there exists $V(x) \in \Sigma(x), W(x) \in \Lambda(B)$ satisfying $(V(x) \times W(x)) \cap \pi K = \emptyset$. There is a finite $A^* \subset A$ with $A \subset \text{int} \pi \left(\bigcup_{A^*} V(x) \right) = V$. Let $W = \bigcap_{A^*} W(x)$. Then V, W have the required properties.

Definition 13. A relation (or a multifunction) λ from X to Y is a function $\lambda : X \rightarrow 2^Y - \{\emptyset\}$. A relation λ from a space

X to a space Y is *upper semicontinuous (u.s.c.)* if for every $W \in \Sigma \lambda(x)$ there is a $V \in \Sigma(x)$ such that $\lambda(V) \subset W$. A multifunction λ from a space X to a space Y has a π -strongly closed graph if $\pi\lambda(\Sigma(x)) = \lambda(x)$ for each $x \in X$.

Theorem 26 If λ is an u.s.c. multifunction on X and π is an adherence dominator then $\pi\lambda\Sigma\{x\} = \pi\lambda\{x\}$.

Proof Clearly $\pi\lambda\{x\} \subset \pi\lambda(\Sigma\{x\})$ and for each $W \in \Sigma \lambda(x)$, some $V \in \Sigma\{x\}$ satisfies $\lambda(V) \subset W$, since λ is u.s.c. and thus $\pi\lambda\Sigma\{x\} \subset \pi\lambda\{x\}$.

Corollary 11. An u.s.c. multifunction λ has a π strongly-closed graph if and only if λ has π closed point images.

It is established for θ -closures that if x and y are points in a space X , then $y \in cl_\theta(x)$ if and only if $x \in cl_\theta(y)$ [5]. The following definition is modelled after this property of θ -closures.

Definition 14. We say that x is equivalent to y if $\pi x = \pi y$ and use the notation $x \equiv y$. For the adherence dominators being considered, $x \in \pi y$ if and only if $y \in \pi x$. If $v \in X$, a π space, then $\pi \bigcap_{\Sigma\{v\}} W = \{v\}$.

In a π space the π closure of each point is trivially compact and maximal in the set of π closures of points ordered by inclusion. We use the last theorem to prove that in any space, the π closures of points satisfy a maximally condition, when the π closure of some point is compact.

Theorem 27 Let Y be a space and let $y_0 \in Y$ with πy_0 compact. Then there is a $y \in Y$ such that (1) $\pi y_0 \subset \pi y$ and (2) πy is maximal in the set of π closures of points when this set is ordered by inclusion.

Proof Let $X = \{y \in Y : \pi y_0 \subset \pi y\}$. For each $y \in X$ we have $y \in \pi y_0$. Moreover if $v \in \mathcal{A}X$ and $W \in \Sigma\{v\}$ then some $y \in W$ satisfies $\pi y_0 \subset \pi y \subset \pi W$. Hence $\pi y_0 \subset \pi v$. So, $v \in X$ and X is closed in Y . Moreover, for each $y \in X, y \in \pi y_0$ since $\pi y_0 \subseteq \pi y$. That is, $y_0 \in \bigcap_X \pi y$. Hence if $y \in X, y \in \pi y_0$ and πy_0 is compact. Therefore X is a compact subset of Y . Since the identity function from X to Y is u.s.c., the proof is complete, in view of Theorem 11 (1) \Rightarrow (2) [4].

5. π Components Relative to a Space

A π component relative to a space X is a π connected relative to X subset which is not properly contained in any π connected relative to X subset. Since πA is π connected relative to X whenever A is π connected relative to X , it follows that a π component relative to X is π closed. It is easy to see that each π connected relative to a space X is contained in a π component relative to X . It is not difficult to see that if H, K are two distinct π components relative to a space X then either $H \cap \pi K = \emptyset$ or $K \cap \pi H = \emptyset$. If $\emptyset \neq A \subset X$ and if $x, y \in A$, we say that x and y are equivalent if whenever (P, Q) is a π separation of A relative to X , $\{x, y\} \subset P$ or $\{x, y\} \subset Q$. If Ax represents the equivalence class of x with respect to this relation on A , call Ax a π quasicomponent of A relative to X . A π quasicomponent of A relative to A is a quasicomponent of A . If $A \subset X$ and $x \in A$ let $\pi S(A, x) = \{P \subset X : \text{for some } Q \subset$

X , (P, Q) is a π separation of A relative to X , satisfying $A = P \cup Q$, and $x \in P$. The following proposition of Ax follows easily.

Proposition 2. If X is a space and $\emptyset \neq A \subset X$, then for each $x \in A$, $Ax = \bigcap_{S(A,x)} P$.

Proposition 3. If X is a space and $\emptyset \neq B \subset A \subset X$, then $Bx \subset Ax$ for each $x \in B$.

Proof Let $x \in B$, $y \in Bx$, (P, Q) a π separation of A relative to X . If $P \cap B = \emptyset$ or $(Q \cap B = \emptyset)$, then $\{x, y\} \subset Q$ or $(\{x, y\} \subset P)$. Otherwise, $(P \cap B, Q \cap B)$ is a π separation of B relative to X . So $\{x, y\} \subset Q \cap B$ (or $\{x, y\} \subset P \cap B$). Therefore, $y \in Ax$.

The following improves the well-known fact that quasicomponents are closed.

Proposition 4. If X is a space and $\emptyset \neq A \subset X$, then Ax is π closed in A for each $x \in A$.

Proof Let $x \in A$, $y \in A - Ax$. Then there is a π partition (P, Q) of A relative to X , $y \in P$, $Ax \subset Q$. Hence $y \notin \pi Ax$ in X . It follows that $y \notin \pi Ax$ in A .

Corollary 12. In a π closed space X , π quasi components are π rigid.

Proof Follows from the fact that in a π space, π closed sets are π rigid.

Theorem 28 If X is a space and A is a non-empty subset of X , then Ax is a π component of A relative to X for each $x \in A$ for which Ax is π connected relative to X .

Proof Suppose that $x \in A$ and Ax is π connected relative to X . Let $B \subseteq A$ be a π connected relative to X and suppose that $Ax \subset B$. Let $y \in B$ and $A = P \cup Q$, where (P, Q) is a π separation relative to X for A . Then $B \subset P$ or $B \subset Q$. So, either $\{x, y\} \subset P$ or $\{x, y\} \subset Q$. Hence $y \in Ax$. Hence Ax is a π component of relative to X , since, thus, Ax π connected relative to X .

Corollary 13. π connected π quasicomponents are π components.

We conclude this article with the following result which states that in a π C-compact space X , each π quasicomponent relative to X of a π closed set A is a π component relative to X of A .

Theorem 29 Let X be π C-compact and let $A \subset X$ be a non-empty π closed set. Then each π quasicomponent Ax of A relative to X is a π component of A relative to X .

Proof Let $A \subset X$ be non-empty and π closed and suppose that $x \in A$. We shall show that Ax is π connected relative to X and then in view of Theorem 28, Ax will be a π component of A relative to X . Let $y \in Ax$ and let $\Omega = \{B \subset A \text{ such that } B \text{ is } \pi\text{closed in } X, x \in B, y \in Bx\}$. Then $\Omega \neq \emptyset$ since $A \in \Omega$. Order Ω by set inclusion and let Ω^* be a chain in Ω . Let $C = \bigcap_{\Omega^*} F$. Since each $F \in \Omega^*$ is π closed and $x \in F$ for each $F \in \Omega^*$, $x \in C$. Let (P, Q) be a π separation relative to X of C where both P and Q are π closed in X and $x \in P$, $y \in Q$. Since X is π C-compact and P and Q are two disjoint π closed sets, in view of Theorem 19, there are disjoint open sets V and W containing P and Q respectively with $\pi V \cap \pi W = \emptyset$. Since $C \subset V \cup W$, there is an $F_0 \in \Omega^*$ such that $F_0 \subset (V \cup W)$. This gives a π separation for F_0 as

$F_0 = (F_0 \cap V) \cup (F_0 \cap W)$, $x \in (F_0 \cap V)$, $y \in (F_0 \cap W)$. This is a contradiction since $F_0 \in \Omega$. Therefore, C is a lowerbound for Ω and hence by Zorn's Lemma, has a minimal element, say C_0 . To show that C_0 is π connected relative to X , suppose that (P, Q) is a π separation relative to X with $C_0 = P \cup Q$. Assume that $P \neq C_0$ and that $x, y \in P$. Since P is π closed in X , there exist P_1 and P_2 such that (P_1, P_2) is a π separation for P relative to X , with $x \in P_1$ and $y \in P_2$. This gives a π separation relative to X for C_0 with $C_0 = P_1 \cup (P_2 \cup Q)$, and $x \in P_1$, $y \in (P_2 \cup Q)$. This is a contradiction since $C_0 \in \Omega$. Thus C_0 is π connected relative to X and $C_0 = Cx$ and $\{x, y\} \subset C_0 \subset Ax$. Therefore, in view of Theorem 5 (2), Ax is π connected relative to X . Hence in view of Theorem 28, Ax is a π component relative to X .

6. Conclusion

Connectedness of a set in a topological space depends on how close are the points of that set to each other and hence depends on the closure operator of that space. The adherence dominator concepts subsumes different closure operators. So, this article brings the study of different topological concepts such as continuity, convergence, compactness and their generalizations in a unified frame work. Recently, the authors have studied and compiled several topological properties using the adherence dominator operator [14].

Every attempt is made to give the appropriate citation whenever a result or concept is used from a source. Connectedness, compactness, maximality conditions using the Zorn's lemma, continuous function, multifunctions, regularly open sets, regularly closed sets, semiregularization etc are concepts and results which could be found and have been in literature on general topology and related topics and hence when these are stated, no particular citation is given. However we do not claim authorship for them.

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