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General Solutions of Some Complex Third-order Differential Equations

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Abstract: According to the Nevanlinna theory, many researches have undertaken the behaviors of meromorphic solutions of complex ordinary differential equations (ODEs). Most of these researches have concentrated on the value distribution and growth of meromorphic solutions of ODEs. However, the existence of a meromorphic general solution is often used as a way to identify equations that are integrable. Especially, the existence of global meromorphic solutions of differential equation $f^{''}+A(z)f=0$ with entire coefficient can be settled, resulting in the characterization of Schwarzian derivatives. This is concerning with the linearly independent solutions of linear differential equations $f^{''}+\frac{h(z)}{(z-z_0)^2}f=0$. The purpose of this present paper is to find explicit solutions of differential equation in terms of finite combinations of known functions, that is, we use local series methods and reduction of order to solve all linearly independent solutions of some third-order ODEs $f^{'''}+\frac{h(z)}{(z-z_0)^3}=0$ with entire coefficient $h(z)$ in the neighborhood of z_0.

Keywords: Ordinary Differential Equation, Local Series Method, Linearly Independent, Meromorphic General Solution

1. Introduction

Ordinary differential equations(ODEs) in the complex domain is an area of mathematics admitting several ways of approach, which basic results can be found in a large number of text-books of differential equations, see, e.g. [12, 13, 16]. At present, many researches focus our interest on Nevanlinna theory, and have undertaken the value distribution of meromorphic solutions of ODEs, see, e.g. [2-8, 11,13-15, 17-18].

However, finding explicit solutions of ODEs in terms of finite combinations of known functions is more difficult. However, it was observed in the late nineteenth and early twentieth centuries that ODEs whose general solutions are meromorphic appear to be integrable in that they can be solved explicitly or they are the compatibility conditions of certain types of linear problems [1]. The condition that the general solution is meromorphic can be replaced by the condition that the ODE possesses the Painlvé property, that is, all solutions are single-valued about all movable singularities.

Finite order functions have special properties and so they have been the subject of intense study [10]. The major result concerning the order of growth of meromorphic solutions of first order ODEs is the following theorem due to Gol'dberg.

Theorem 1.1. [6] All meromorphic solutions of the first order ODE

$$\Omega(z,f,f^{'})=0, \tag{1}$$

where Ω is polynomial in all its arguments, are of finite order.

A generalization of Gol'dberg's result to second order algebraic equations have been conjectured by Bank [2]. Hayman [9] described a further generalization of Bank's conjecture to nth-order ODEs. If $f(z)$ is a meromorphic solution of

$$\Omega(z,f,f^{'},\cdots,f^{(n)})=0, \tag{2}$$

where Ω is polynomial in $z,f,f^{'},\cdots,f^{(n)}$, then we have

$$T(r,f)<a\exp_{n-1}(br^c),\quad 0\le r<+\infty, \tag{3}$$

where a, b and c are constants and $\exp_j(x)$ is defined by

$$\exp_0(x) = x, \ \exp_1(x) = e^x,$$
$$\exp_j(x) = \exp\{\exp_{j-1}(x)\}.$$

In this paper, we will focus our interest on finding explicit solutions of differential equation in terms of finite combinations of known functions, that is, we use local series methods and reduction of order to solve all linearly independent solutions of some third-order differential equations.

The remainder of the paper is organized as follows. In Section 2, we recalled some results on the existence of global meromorphic solutions of second-order ODE

$$f^{''} + \frac{h(z)}{(z-z_0)^2} f = 0,$$

which resulted in the characterzation of Schwarzian derivatives. In Section 3, the explicit solutions of differential equation in terms of finite combinations of known functions to solve some third-order ODEs

$$f^{'''} + \frac{h(z)}{(z-z_0)^3} = 0$$

with entire coefficient $h(z)$ in the neighborhood of z_0 have been arrived.

2. Explicit Solutions of Second Order Differential Equation

When considering the formal form of second order differential equation

$$f^{''} + A(z) f = 0, \tag{4}$$

where $A(z)$ is meromorphic, we need first to find out whether its meromorphic solutions exist or not. The existence of global meromorphic solutions of (4) can be settled, resulting in the characterzation of Schwarzian derivatives, see Theorem 2.1 and Corllary 2.2 obtained by Herold[12].

Theorem 2.1. [12] Let $G \subset \mathbb{C}$ be a simply connected domain, such that $A(z)$ is meromorphic in G. The quotient of any two local solutions of (4) is meromorphic and admits a meromorphic continuation into the whole G if and only if at all poles of $A(z)$, the Laurent expansion of $A(z)$ around z_0 has the form

$$A(z) = \frac{b_0}{(z-z_0)^2} + \frac{b_1}{z-z_0} + b_2 + ..., \tag{5}$$

where

$$4b_0 = 1 - m^2, m \; is \; integer \; and \; m \geq 2, \tag{6}$$

and

$$D(z_0) = \begin{vmatrix} 1-m & 0 & \cdots & 0 & b_1 \\ b_1 & 4-2m & \cdots & 0 & b_2 \\ b_2 & b_1 & \ddots & 0 & b_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m-2} & b_{m-3} & \cdots & (m-1)^2-(m-1)m & b_{m-1} \\ b_{m-1} & b_{m-2} & \cdots & b_1 & b_m \end{vmatrix} = 0. \tag{7}$$

Moreover, these continuations g of local quotients all satisfy, in G,

$$S_g := \left(\frac{g^{''}}{g^{'}}\right)^{'} - \frac{1}{2}\left(\frac{g^{''}}{g^{'}}\right)^2 = 2A(z).$$

Corollary 2.1. [12] Let $G \subset \mathbb{C}$ be a simple connected domain such that $A(z)$ is meromorphic in G. The differential equation (4) admits two linearly independent meromorphic solutions in G if and only if at all poles z_0 of $A(z)$, the Laurent expansion of $A(z)$ is of the form (5), satisfying (6) with an odd integer $m \geq 3$ and (7).

In order to prove Theorem 2.1, Herold first gave out explicit solutions of equation

$$f^{''} + \frac{h(z)}{(z-z_0)^2} f = 0, \tag{8}$$

where $h(z)$ is analysis in $|z - z_0| < R, R > 0$. Obviously, (8) is just a simplified form of (4) and satisfies (5) and some special conditions. He decared

Theorem 2.2. [12] Suppose $h(z)$ is analytic in $|z - z_0| < R$, and consider the differential equation (8) in the disc $|z - z_0| < R$. Let ρ_1, ρ_2 be the roots of

$$\rho(\rho - 1) + h(z_0) = 0,$$

assuming that $\rho_1 - \rho_2 \in \mathbb{Z}\backslash\{0\}$. Denote by $D = D(r)$ the slit disc

$$D := \{z||z - z_0| < r\}\backslash\{z_0 + t|0 \leq t < r\}.$$

Then (8) admits, in some slit disc $D = D(r), r \leq R$, two linearly independent solutions f_1, f_2 of the following form

$$\begin{cases} f_1 = (z-z_0)^{\rho_1} \sum_{i=0}^{\infty} a_i (z-z_0)^i, a_0 \neq 0, \\ f_2 = k f_1(z) \log(z-z_0) + (z-z_0)^{\rho_2} \sum_{j=0}^{\infty} b_j (z-z_0)^j, \end{cases} \tag{9}$$

where either $k = 0$ or $k = 1$.

3. General Solutions of Third Order Differential Equation

In this section, we discuss about linearly independent solutions of the following third order differential equation.

$$f^{'''} + \frac{h(z)}{(z-z_0)^3} f = 0, \tag{10}$$

where $h(z)$ is analytic $|z - z_0| < R$. We want to find explicit solutions of linear differential equation (10) in terms of finite combinations of known functions, and obtain

Theorem 3.1. Suppose $h(z)$ is analytic $|z - z_0| < R$, and consider the differential equation (10) in the disc $|z-z_0| < R$. Let ρ_1, ρ_2, ρ_3 be the roots of

$$\rho(\rho-1)(\rho-2) + h(z_0) = 0,$$

assuming that $\rho_i - \rho_j \in \mathbb{Z}\backslash\{0\}, 1 \le i < j \le 3$ and $h(z_0) \neq 0$. Then (10) admits, in some slit disc $D = D(r), r \le R$, three linearly independent solutions f_1, f_2, f_3 of one of forms:

$$\begin{cases} f_1 = (z-z_0)^{\rho_1} \sum\limits_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 = (z-z_0)^{\rho_2} \sum\limits_{i=0}^{\infty} c_i^* (z-z_0)^i, \\ f_3 = (z-z_0)^{\rho_3} \sum\limits_{i=0}^{\infty} c_i^{**} (z-z_0)^i, \end{cases} \tag{11}$$

and

$$\begin{cases} f_1 = (z-z_0)^{\rho_1} \sum\limits_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 = (z-z_0)^{\rho_2} \sum\limits_{i=0}^{\infty} c_i^* (z-z_0)^i, \\ f_3 = \xi_k f_1 \log(z-z_0) \\ \quad + \gamma_k f_1 \int \left(\frac{f_2}{f_1}\right)' \log(z-z_0) dz \\ \quad + (z-z_0)^{-\rho_2-\rho_1+1} \Phi(z), \end{cases} \tag{12}$$

where $\Phi(z)$ is analytic in D.

The idea of the proof is to submit the Laurent series of $f(z)$ and $h(z)$ to (10) and to compare with their coefficients. By this way, we can conclude the indicial equation $\rho(\rho-1)(\rho-2) + h(z_0) = 0$. Theorem 3.1 shows the finite combinations of known functions f_1, f_2 and f_3 when $h(z_0) \neq 0$. If $h(z_0) = 0$, we further obtain

Theorem 3.2. Suppose $h(z)$ is analytic $|z - z_0| < R$, and consider the differential equation (10) in the disc $|z-z_0| < R$. Let ρ_1, ρ_2, ρ_3 be the roots of

$$\rho(\rho-1)(\rho-2) + h(z_0) = 0,$$

assuming that $\rho_i - \rho_j \in \mathbb{Z}\backslash\{0\}, 1 \le i < j \le 3$, and $h(z_0) = 0$. Then except the forms of (11),(12), (10) also admits , in some slit disc $D = D(r), r \le R$, three linearly independent solutions f_1, f_2, f_3 of one of forms:

$$\begin{cases} f_1 &= (z-z_0)^2 \sum_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 &= f_1 \int g_1 dz = k_1 f_1 \log(z-z_0) \\ & \quad + (z-z_0)^{\rho_1-1} \phi_3(z), \\ f_3 &= f_1 \int g_2 dz = k_2 f_1 \log(z-z_0) \\ & \quad + (z-z_0)^{\rho_1+1} \phi_4(z). \end{cases} \tag{13}$$

$$\begin{cases} f_1 &= (z-z_0)^2 \sum_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 &= c_1 f_1 \log(z-z_0) + (z-z_0)^{\rho_1-1} \phi_1(z), \\ f_3 &= d_2 f_1 \log(z-z_0) + (z-z_0)^{\rho_1-2} \phi_1(z). \end{cases} \tag{14}$$

$$\begin{cases} f_1 &= \sum_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 &= f_1 \int g_1 dz = (z-z_0)^{\rho_1+1} \phi_5(z), \\ f_3 &= f_1 \int g_2 dz = (z-z_0)^{\rho_1+1} \phi_6(z). \end{cases} \tag{15}$$

$$\begin{cases} f_1 &= (z-z_0) \sum_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 &= (z-z_0)^{\rho_1+1} \phi_7(z), \\ f_3 &= \xi_1 f_1 \int \Phi(z) \log(z-z_0) + \varsigma_1 f_1 \log(z-z_0) \\ & \quad + (z-z_0)^{\rho_1-1} \phi_8(z). \end{cases} \tag{16}$$

$$\begin{cases} f_1 &= \sum_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 &= (z-z_0)^{\rho_1+2} \phi_9(z), \\ f_3 &= \xi_2 f_1 \int (z-z_0) \Phi(z) \log(z-z_0) dz \\ & \quad + (z-z_0)^{\rho_1+1} \phi_{10}(z). \end{cases} \tag{17}$$

$$\begin{cases} f_1 &= (z-z_0) \sum_{i=0}^{\infty} c_i (z-z_0)^i, \\ f_2 &= (z-z_0)^{\rho_1+2} \phi_{11}(z), \\ f_3 &= \xi_3 f_1 \int (z-z_0) \Phi(z) \log(z-z_0) dz \\ & \quad + \varsigma_3 f_1 \log(z-z_0) \\ & \quad + (z-z_0)^{\rho_1-2} \phi_{12}(z). \end{cases} \tag{18}$$

where $\Phi(z)$ and $\phi_j(z), j = 1, 2, \cdots, 12$ are analytic.

We now give some Lemmas to prove theorems.

The general solutions of differential equation come from the finite combinations of known functions. The number and forms of known functions can detect the forms of solutions. If two known functions are determinate, we have

Lemma 3.1. Suppose that (10) possesses two linearly meromorphic solutions $f_1 = (z-z_0)^{\rho_1} \sum_{i=0}^{\infty} a_i (z-z_0)^i$ and $f_2 = (z-z_0)^{\rho_2} \sum_{i=0}^{\infty} b_i (z-z_0)^i$, satisfying that $\rho_1 \neq \rho_2$.

Then another solution of (10) is of the form

$$\begin{aligned} f_3 = \xi_k f_1 \log(z-z_0) + \gamma_k f_1 \int (\frac{f_2}{f_1})' \log(z-z_0) dz \\ + (z-z_0)^{-\rho_2-\rho_1+1} \Phi(z), \end{aligned} \tag{19}$$

where $\Phi(z)$ is analytic.

Proof. Assume that $f = f_1 F$ is a solution of (10). Then

$$\begin{aligned} f' &= f_1' F + f_1 F, \\ f'' &= f_1'' F + 2 f_1' F' + f_1 F'', \\ f''' &= f_1''' F + 3 f_1'' F' + 3 f_1' F'' + f_1 F'''. \end{aligned}$$

Substituting the above equations into (10), we obtain

$$f_1g'' + 3f_1'g' + 3f_1''g = 0, \tag{20}$$

where $g = F'$.

In order to get f_3, we need to solve the equation (20). Since f_2 is also a solution of (10),we can calculate that $g_1 = \left(\frac{f_2}{f_1}\right)'$ is one solution of (20).

Assume again that $g = g_1G$ is one solution of (20). Then we have

$$g' = g_1'G + g_1G',$$
$$g'' = g_1''G + 2g_1'G' + g_1G''.$$

Substituting the above equations into (20), we obtain

$$(2f_1g_1' + 3f_1'g_1)W = -f_1g_1W', \tag{21}$$

where $W = G'$.

Solve the equation (21) and we have $W = cg_1^{-2}f_1^{-3}$. Substituting W into $g = g_1G$, then $g_2 = g_1\int Wdz = g_1\int cg_1^{-2}f_1^{-3}dz$ is a solution of (20). What's more, let $f_3 = f_1\int g_2dz$ and then f_3 is the solution of (10) that is arrived.

We now calculate the explicit form of f_3. Actually

$$g_1^{-2} = \left[\left(\frac{f_2}{f_1}\right)'\right]^{-2} = (z-z_0)^{-2\rho_2+2\rho_1+2}\Phi_1(z),$$
$$G = c\int g_1^{-2}f_1^{-3}dz = \gamma_k\log(z-z_0)$$
$$+ (z-z_0)^{-2\rho_2-\rho_1+3}\Phi_2(z),$$
$$g_2 = g_1G = \gamma_kg_1\log(z-z_0) + (z-z_0)^{-\rho_2-2\rho_1+2}\Phi_3(z).$$

Hence

$$f_3 = f_1\int g_2dz = \xi_kf_1\log(z-z_0)$$
$$+ \gamma_kf_1\int g_1\log(z-z_0)dz + (z-z_0)^{-\rho_2-\rho_1+3}\Phi(z),$$

where $\Phi_1(z), \Phi_2(z), \Phi_3(z), \Phi(z)$ are analytic.

However, if just one known function is determinate, we have

Lemma 3.2. Suppose that (10) just possesses one solution $f_1 = (z-z_0)^{\rho_1}\sum_{i=0}^{\infty}a_i(z-z_0)^i$. Then all other solutions of (10) admits one of the following forms: (13)−(18).

Proof. Using the same method as in Lemma 3.3, we still need to solve equation (20), while in this case $g(z)$ is unknown. We need to find out a set of linearly independent solutions g_1 and g_2. Then let $f_2 = f_1\int g_1dz$, $f_3 = f_1\int g_2dz$ and such f_2,f_3 are the solutions of (10). Assume that $g(z) = (z-z_0)^k\sum_{i=0}^{\infty}c_i(z-z_0)^i$. In the following, we will split our proofs into six cases.

Case 1. Suppose that $\rho_1 \neq 0, 1$ and $k \neq 0, 1$. Then

$$f_1 = a_0(z-z_0)^{\rho_1} + a_1(z-z_0)^{\rho_1+1} + ...,$$
$$f_1' = a_0\rho(z-z_0)^{\rho_1-1} + a_1(\rho+1)(z-z_0)^{\rho_1} + ...,$$
$$f_1'' = a_0\rho(\rho-1)(z-z_0)^{\rho_1-2} + a_1(\rho+1)\rho(z-z_0)^{\rho_1-1} + ...,$$
$$g = c_0(z-z_0)^k + c_1(z-z_0)^{k+1} + ...,$$
$$g' = c_0k(z-z_0)^{k-1} + c_1(k+1)(z-z_0)^k + ...,$$
$$g'' = c_0k(k-1)(z-z_0)^{k-2} + c_1(k+1)k(z-z_0)^{k-1} + ...$$

Substitute the above equations into (20) and compare the coefficients of the lowest term $(z-z)^{\rho_1+k-2}$, we obtain

$$a_0c_0k(k-1) + 3a_0\rho_1c_0k + 3a_0\rho_1(\rho_1-1)c_0 = 0.$$

It is necessary to notice that $\triangle = -3(\rho_1-1)^2 + 4 \geq 0$ and $\rho_1 \in \mathbb{Z}$, hence $\rho_1 = 2$, $k = -2$ or $k = -3$.

When $\rho_1 = 2, k = -2$, for any $n \in \mathbb{Z}$,

$$f_1 = a_0(z-z_0)^2 + ... + a_n(z-z_0)^{n+2} + ...,$$
$$f_1' = 2a_0(z-z_0) + ... + (n+2)a_n(z-z_0)^{n+1} + ...,$$
$$f_1'' = 2a_0 + ... + (n+2)(n+1)a_n(z-z_0)^n + ...,$$
$$g_1 = c_0(z-z_0)^{-2} + ... + c_n(z-z_0)^{n-2} + ...,$$
$$g_1' = -2c_0(z-z_0)^{-3} + ... + (n-2)c_n(z-z_0)^{n-3} + ...,$$
$$g_1', = 6c_0(z-z_0)^{-4} + ... + (n-2)(n-3)c_n(z-z_0)^{n-4} + ...$$

Substitute the above equations into (20) and consider the coefficients of the lowest term $(z-z_0)^{n-2}$, we have

$$[a_0(n-2)(n-3)c_n + ... + 6a_nc_0]$$
$$+ 3[2a_0(n-2)c_n + ... + (n+2)a_n(-2c_0)]$$
$$+ 3[2a_0c_n + ... + (n+2)(n-1)a_nc_0] = 0,$$

and then

$$c_1 = -\frac{3a_1c_0}{2a_0},$$
$$c_2 = -\frac{11a_1c_1 + 18a_2c_0}{6a_0},$$
$$...,$$
$$c_n = \frac{\sum_{i=1}^{k}K_i(n)a_ic_{n-i}}{n(n+1)a_0},$$
$$...,$$

where

$$K_i(n) = (n-i-2)(n-i-3) + 3(i+2)(n-i-2)$$
$$+ 3(i+2)(i+1) \leq 7(n+2)^2.$$

We now affirm that the formal power series $g_1 = (z-z_0)^{-2}\sum_{i=0}^{\infty}c_i(z-z_0)^i$ converges. If we can prove that for some $r \in (0, R)$ and some $M > 0$,$|c_i|r^i \leq M$ holds for $i = 0, 1, 2, ...$, we have $\limsup|c_i|^{\frac{1}{i}} \leq \frac{1}{r}$ and therefore $g_1(z)$ converges. Actually, suppose that there exists some $r > 0, M > 0$ such that $|c_i|r^i \leq M$ for $i = 0, 1, ..., n-1$. Since $(z-z_0)^{\rho_1}\sum_{i=1}^{\infty}a_i(z-z_0)^i$ converges and vanishes at $z = z_0$, decreasing r if needed, we have for each n, $\sum_{i=0}^{\infty}|a_i|r^i \leq \frac{(n+1)|a_0|}{7(n+2)^2}$. Then for $i = n$,

$$|c_n|r^n \leq \frac{\sum_{i=1}^{n} | K_i || a_i | r^i | c_{n-i} | r^{n-i}}{n(n+1) | a_0 |} \leq M.$$

Hence $g_1 = (z-z_0)^{-2}\sum_{i=0}^{\infty} c_i(z-z_0)^i$ converges and is one solution of (20). Using the same method we can prove that when $\rho_1 = 2, k = -3$, $g_2 = (z-z_0)^{-3}\sum_{i=0}^{\infty} di(z-z_0)^i$ is also a solution of (20). Therefore, we obtain the other solutions of (10) as follows:

$$f_2 = f_1 \int g_1 dz = c_1 f_1 \log(z-z_0) + (z-z_0)^{\rho_1 - 1}\phi_1(z),$$
$$f_3 = f_1 \int g_2 dz = d_2 f_1 \log(z-z_0) + (z-z_0)^{\rho_1 - 2}\phi_1(z).$$

Case 2. Suppose that $\rho_1 \neq 0, 1$ and $k = 0$. Then by comparing the coefficients of the term $(z-z)^{\rho_1 - 2}$ in (20) we obtain

$$a_0\rho_1(\rho_1 - 1)c_0 = 0.$$

Therefore $\rho_1 = 0$ or $\rho_1 = 1$, a contradiction.

Case 3. Suppose that $\rho_1 \neq 0, 1$ and $k = 1$. Then by comparing the coefficients of the term $(z-z)^{\rho_1 - 1}$ in (20) we obtain

$$3a_0\rho_1 c_0 + 3a_0\rho_1(\rho_1 - 1)c_0 = 0.$$

Therefore $\rho_1 = 0$, a contradiction.

Case 4. Suppose that $\rho_1 = 0$ and $k \neq 0, 1$. Then by comparing the coefficients of the term $(z-z_0)^{k-2}$ in (20) we obtain

$$a_0 c_0 k(k-1) = 0.$$

Therefore $k = 0$ or $k = 1$, a contradiction.

Case 5. Suppose that $\rho_1 = 1$ and $k \neq 0, 1$. Then by comparing the coefficients of the term $(z-z_0)^{k-1}$ in (20), we obtain

$$a_0 c_0 k(k-1) + 3a_0 c_0 k = 0.$$

Therefore $k = -2$. In this case $g_1 = (z-z_0)^{-2}\sum_{i=0}^{\infty} s_i(z-z_0)^i$ is a solution of (20). By Lemma 3.3, another solution of (20) is

$$g_2 = g_1 G = g_1 \int g_1^{-2} f_1^{-3} dz = \Phi_3(z).$$

Hence, we have three linearly dependent solutions of (10) as follows:

$$f_1 = (z-z_0)\sum_{i=0}^{\infty} a_i(z-z_0)^i,$$
$$f_2 = f_1 \int g_1 dz = k_1 f_1 \log(z-z_0) + (z-z_0)^{\rho_1 - 1}\phi_3(z),$$
$$f_3 = f_1 \int g_2 dz = (z-z_0)^{\rho_1 + 1}\phi_4(z).$$

Case 6. Suppose that $\varrho_1 = 0$ and $k = 0$. Then by comparing the coefficient of the term $(z-z_0)^n$ in (20), we obtain the form of common term

$$c_2 = -\frac{3a_1c_1 + 3a_2c_1}{2a_0},$$
$$c_3 = -\frac{8a_1c_2 + 12a_2c_1 + 18a_3c_0}{6a_0},$$
$$\ldots,$$
$$c_n = -\frac{\sum_{i=1}^{n} K_i(n)c_i a_{n-i}}{n(n-1)a_0}.$$

Here c_0, c_1 is determined arbitrarily.

Using the same method as in Case 1, we can prove that the formal power series $g_1 = \sum_{i=0}^{\infty} c_i(z-z_0)^i$ converge. By Lemma 3.3,

$$g_2 = g_1 G = g_1 \int g_1^{-2} f_1^{-3} dz = \sum_{i=0}^{\infty} \xi_i(z-z_0)^i.$$

Hence

$$f_2 = f_1 \int g_1 dz = (z-z_0)^{\rho_1+1}\phi_5(z),$$
$$f_3 = f_1 \int g_2 dz = (z-z_0)^{\rho_1+2}\phi_6(z).$$

Similarly,
for $\rho_1 = 1, k = 0$,

$$f_2 = (z-z_0)^{\rho_1+1}\phi_7(z),$$
$$f_3 = \xi_1 f_1 \int \Phi(z)\log(z-z_0) + \varsigma_1 f_1 \log(z-z_0) + (z-z_0)^{\rho_1-1}\phi_8(z).$$

for $\rho_1 = 0, k = 1$,

$$f_2 = (z-z_0)^{\rho_1+2}\phi_9(z),$$
$$f_3 = \xi_2 f_1 \int (z-z_0)\Phi(z)\log(z-z_0)dz + (z-z_0)^{\rho_1+1}\phi_{10}(z).$$

for $\rho_1 = 1, k = 1$,

$$f_2 = (z-z_0)^{\rho_1+2}\phi_{11}(z),$$
$$f_3 = \xi_3 f_1 \int (z-z_0)\Phi(z)\log(z-z_0)dz + \varsigma_3 f_1 \log(z-z_0) + (z-z_0)^{\rho_1-2}\phi_{12}(z).$$

Remark 3.1. It is necessary to realize that, in Lemma 3.4, all cases of solutions satisfy additional condition $h(z_0) = 0$.

Based on the above lemmas, we can prove Theorem 3.1 and 3.2.

Proof. of Theorem 3.1. Suppose that

$$f(z) = (z-z_0)^{\rho}\sum_{i=0}^{\infty} c_i (z-z_0)^i,$$

and

$$h(z) = \sum_{i=0}^{\infty} \beta_i (z-z_0)^i.$$

Then we conclude that

$$\begin{aligned} f'''(z) &= c_0\rho(\rho-1)(\rho-2)(z-z_0)^{\rho-3} \\ &\quad + c_1(\rho+1)\rho(\rho-1)(z-z_0)^{\rho-2} + \dots \end{aligned}$$

Substituting the above into (10), we obtain

$$\begin{aligned} &c_0\varphi_0(\rho) = 0, \\ &c_1\varphi_0(\rho+1) + c_0\varphi_1(\rho) = 0, \\ &\dots, \\ &c_n\varphi_0(\rho+n) + c_{n-1}\varphi_1(\rho) + \dots + c_1\varphi_{n-1}(\rho) \\ &\qquad + c_0\varphi_n(\rho) = 0, \\ &\dots, \end{aligned} \tag{22}$$

where

$$\begin{cases} \varphi_0(\rho) = \rho(\rho-1)(\rho-2) + h(z_0), \\ \varphi_i(\rho) = \beta_i. \end{cases}$$

As ρ_1, ρ_2, ρ_3 are distinct roots of $\rho(\rho-1)(\rho-2)+h(z_0) = 0$, without loss of generalization, we may assume that $\rho_1 > \rho_2 > \rho_3$.

When $\rho = \rho_1$, for any $k \in \mathbb{Z}$, $\varphi_0(\rho_1 + k) \neq 0$, then

$$c_k = -\frac{c_{k-1}\beta_1 + \dots + c_0\beta_k}{\varphi_0(\rho_1+k)}.$$

Besides we need to prove that the formal power series $f_1 = (z-z_0)^{\rho_1}\sum_{i=0}^{\infty} c_i(z - z_0^i)$ converges. Actually assume that $|c_i|r^i \le M$ for $i = 0, 1, \dots, n-1$, and we need to prove that such inequality still holds for $i = n$.By (22) we have

$$|\varphi_0(\rho+n)||c_n| \le \sum_{i=1}^{n} |c_{n-i}||\beta_i|.$$

By simple calculation we have

$$\begin{aligned} \varphi_0(\rho+n) &= \varphi_0(\rho+n) - \varphi_0(\rho) \\ &= (\rho+n)(\rho+n-1)(\rho+n-2) \\ &\quad + \beta_0 - \rho(\rho-1)(\rho-2) - \beta_0. \end{aligned}$$

Hence $|\varphi_0(\rho+n)| \ge cn^3$.

Therefore

$$|\varphi_0(\rho+n)||c_n|r^n \leq \sum_{i=1}^{n} |c_{n-i}|r^{n-i}|\beta_i|r^i$$
$$\leq M \sum i = 1^n |\beta_i| r^i \leq Mck.$$

Then
$$|c_n r^n| \leq \frac{Mck}{cn^3} \leq M.$$

When $\rho = \rho_2$, there exists $k_1 = \rho_1 - \rho_2$ such that $\varphi_0(\rho_2 + k_1) = 0$. If

$$c^*_{k_1-1}\beta_1 + ... + c^*_0\beta_{k_1} = 0, \tag{23}$$

then we may determine $c^*_{k_1}$ arbitrarily and for $k \neq k_1$,

$$c^*_k = -\frac{c^*_{k-1}\beta_1 + ... + c^*_0\beta_k}{\varphi_0(\rho_2 + k)}.$$

Therefore $f_2 = (z - z_0)^{\rho_2} \sum_{i=0}^{\infty} c^*_i (z - z_0^i)$ converges as a solution of (10). However if (23) doesn't hold, we cannot find out the form of f_2 in this way.

When $\rho = \rho_3$, there exists $k_2 = \rho_2 - \rho_3$ and $k_3 = \rho_1 - \rho_3$ such that $\varphi_0(\rho_2 + k_i) = 0, i = 2, 3$. If

$$c^{**}_{k_2-1}\beta_1 + ... + c^*_0\beta_{k_2} = 0 \tag{24}$$

as well as
$$c^{**}_{k_3-1}\beta_1 + ... + c^{**}_0\beta_{k_3} = 0 \tag{25}$$

holds, we may determine $c^{**}_{k_2}$ and $c^{**}_{k_3}$ arbitrarily and for $k \neq k_2, k_3$,
$$c^{**}_k = -\frac{c^{**}_{k-1}\beta_1 + ... + c^{**}_0\beta_k}{\varphi_0(\rho_3 + k)}.$$

Therefore $f_3 = (z - z_0)^{\rho_3} \sum_{i=0}^{\infty} c^{**}_i (z - z_0^i)$ converges as a solution of (10). However if either (24) or (25) does not hold, we cannot find out the form of f_3 in this way. Thus, we need split our proofs into three cases.

Case i. When (23),(24) and (25) hold, we can immediately find out the form of f_1, f_2, f_3 as (11).

Case ii. When (23) holds, either (24) or (25) doesn't hold, then f_1, f_2 are known solutions of (10) and by Lemma 3.3 we can find out the form of f_1, f_2, f_3 as (12).

Case iii. When none of (23), (24), (25) holds, then f_1 is the only known solution of (10). In this case, $h(z_0) \neq 0$, so that $\rho_1 \neq 0, 1, 2$. By Lemma 3.4, we know that in this case we cannot find out suitable form of f_1, f_2, f_3.

Proof. of Theorem 3.2. Using the smilar method as in Theorem 3.1, except Case i and Case ii hold, we further deduce from Lemm 3.4 that one of the forms of f_1, f_2, f_3 as (13)−(18) holds. In this case, $h(z_0) = 0$, none of (23) , (24),(25) holds, and f_1 is the only known solution of (10).

For a special case, we also obtain

Theorem 3.3. Suppose $h(z)$ is analytic $|z - z_0| < R$, and consider the differential equation (10) in the disc $|z - z_0| < R$. Let ρ_1, ρ_2, ρ_3 be the roots of

$$\rho(\rho-1)(\rho-2) + h(z_0) = 0,$$

assuming that $\rho_1 = \rho_2$ while $\rho_2 \neq \rho_3$, $\rho_i \in \mathbb{Z}, i = 1, 2, 3$ and $h(z_0) \neq 0$.

Then (10)admits three linearly independent solutions f_1, f_2, f_3 of the following forms

$$\begin{cases} f_1 = (z-z_0)^{\rho_1} \sum_{i=0}^{\infty} a_i (z-z_0)^i \\ f_2 = (z-z_0)^{\rho_2} \sum_{i=0}^{n} b_i (z-z_0)^i \\ f_3 = \xi_k f_1 \log(z-z_0) + \gamma_k f_1 \int \left(\frac{f_2}{f_1}\right)' \log(z-z_0) dz \\ \quad + (z-z_0)^{-\rho_3-\rho_1+1} \Phi(z) \end{cases}$$

Proof. Assume that $\rho_1 > \rho_3$. Similarly as in the proof of Theorem 3.1, $f_1 = (z - z_0)^{\rho_1} \sum_{i=0}^{\infty} a_i (z - z_0)^i$ is a solution of (10).

Let $k = \rho_3 - \rho_1$, then if $c^*_k\beta_1 + ... + c_0\beta_k = 0$, we have

$$f_2 = (z-z_0)^{\rho_2} \sum_{i=0}^{n} b_i (z-z_0)^i.$$

By Lemma 3.3, we have

$$f_3 = \xi_k f_1 \log(z-z_0) + \gamma_k f_1 \int \left(\frac{f_2}{f_1}\right)' \log(z-z_0) dz$$
$$+ (z-z_0)^{-\rho_3-\rho_1+1} \Phi(z).$$

If $c^*_k\beta_1 + ... + c_0\beta_k \neq 0$, which means that we cannot find out the form of f_2 in this way. As $h(z_0) \neq 0$, by Lemma 3.4, we cannot find the solutions of (10).

Remark 3.2. Suppose $h(z)$ is analytic $|z - z_0| < R$, and consider the differential equation (10) in the disc $|z - z_0| < R$. Let ρ_1, ρ_2, ρ_3 be the roots of

$$\rho(\rho-1)(\rho-2) + h(z_0) = 0$$

assuming that $\rho_1 = \rho_2 = \rho_3 \in \mathbb{Z}$. Though $h(z_0) \neq 0$ in this case, we cannot calculate out the explicit forms of solutions of (10) by Lemma 3.3 and 3.4.

4. Conclusion and Further Discussion

It is well known that every holomorphic function on a simply connected domain in the complex plane can be realized as the Schwarzian derivative of a function that is meromorphic on a given domain. Furthermore, This function is essentially unique by a Möbius transformation. Thus, various results about solutions to second order differential equations with meromorphic coefficients are related to this theme.

In this paper, our main result are concerned with a very particular type of a third order differential equation (10). We use local series methods and reduction of order to solve all linearly independent solutions of some third-order ODEs (10). Thus, the explicit solutions of differential equation (10) in terms of finite combinations of known functions.

Throughout our paper, our results are raised from a very natural question. Some profound questions should be further discussed. Second order ODE of (8) has connection to Teichmuller theory. But, when n is greater than or equal to 3, we do not know whether there is connections with Teichmuller theory or not. Similar results hold if we take an n-th order differential equations of the same type to (10). It is more complicated for us to detect all linearly independent solutions of some n-order ODEs by using local series methods and reduction of order. We need to use computer technology on a large scale.

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References

[1] Ablowitz, M. J., and P. A. Clarkson.*Soliton, nonlinear evolution equations and inverse scattering*, London Mathematical Society Lecture Note Series, 149, Cambridge University Press, Cambridge, 1991.

[2] Bank, S. B. *Some results on analytic and meromorphic solutions of algebraic differential equations*, Advances in Math., 15 (1975): 41–62.

[3] Chiang, Y. M. and Halburd R. G., *On the meromorphic solutions of an equation of Hayman*, J. Math. Anal. Appl., 281 (2) (2003), 663–677.

[4] Chen, W., Wang, Q. and Yuan, W., *Meromorphic solutions of two certain types of nonlinear differential equations*, Rocky Mountain J. Math., 50 (2) (2020), 479–497.

[5] Dilip Chandra, P.,Jayuanta, R. and Kapil, R., *On the growth of solutions of some non-linear complex differential equations*, Korean J. Math., 28 (2) (2020), 295–309.

[6] Gol'dberg, A. A. *On single valued solution of first order differential equations(in Russian)*, Ukrain. Mat. Zh., 8, (1956), 254–261.

[7] Halburd, R. and Korhonen, R., *Growth of meromorphic solutions of delay differential equations*, Proc. Amer. Math. Soc., 145 (6) (2017), 2513–2526.

[8] Halburd, R. and Wang, J., *All admissible meromorphic solutions of Hayman's equation*, Int.Math.Res.Not., 2015 (18) (2015), 8890–8902.

[9] Hayman, W. K., *The growth of solutions of algebraic differential equations*, Mat.Appl., 7 (2) (1996), 67–73.

[10] Hayman, W. K., *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[11] Heittokangas, J., Ishizaki, K., Laine, I. andTohge, K., *Complex oscillation and nonnscillation results*, Trans. Amer. Math. Soc., 372 (9) (2019), 6161–6182.

[12] Herold, H. *Differentialglechungen im Komplexen*, Vandenhoeck Ruprecht, Göttngen, 1975.

[13] Laine, I., *Nevanlinna theory and complex differental equations*, de Gruyter, Berlin, 1993.

[14] Long, J., Heittokangas, J., Ye, Z., *On the relationship between the lower order of coefficients and the growth of solutions of differential equations*, J. Math. Anal. Appl., 444 (1) (2016), 153–166.

[15] Lü, F., Lü, W., Li, C. and Xu, J., *Growth and uniqueness related to complex differential and difference equations*, Results Math., 74 (1) (2019), 1–18.

[16] Matsuda, M. *First order algebraic differential equations*, Lecture Notes in Math.,804, Springer, Berln, 1980.

[17] Mokhon'ko, A. Z., and Kolyasa, L. I., *Some properties of meromorphic solutions of linear differential equation with meromorphic coefficients*, Mat. Stud., 52 (2) (2019), 166–172.

[18] Zhang, R. R. and Huang, Z. B., *Entire solutions of delay differential equations of Malmquist type*, J. Appl. Anal. Comput., 10 (5) (2020), 1720–1740.