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Similarity Solution of (2+1)-Dimensional Calogero-Bogoyavlenskii-Schiff Equation Lax Pair

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Abstract: In this paper, we discussed and studied the solutions of the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation. The Calogero-Bogoyavlenskii-Schiff equation describes the propagation of Riemann waves along the y-axis, with long wave propagating along the x-axis. Lax pair and Bäcklund transformation of the Calogero-Bogoyavlenskii-Schiff equation are derived by using the singular manifold method (SMM). The optimal Lie infinitesimals of the Lax pair are obtained. The detected Lie infinitesimals contain eight unknown functions. These functions are optimized through the commutator table. The eight unknown functions are evaluated through the solution of a set of linear differential equations, in which solutions lead to optimal Lie vectors. The CBS Lax pair is reduced by using the optimal Lie vectors to a system of ordinary differential equations (ODEs). The solitary wave solutions of Calogero-Bogoyavlenskii-Schiff equation Lax pair's show soliton and kink waves. The obtained similarity solutions are plotted for different arbitrary functions and compared with previous analytical solutions. The comparison shows that we derive new solutions of Calogero-Bogoyavlenskii-Schiff equation by using the combination of two methods, which is different from the previous findings.

Keywords: Calogero-Bogoyavlenskii-Schiff Equation, Singular Manifold Method, Lax Pair, Lie Infinitesimals, Similarity Solutions

1. Introduction

Derivation of the Lax pairs of a nonlinear partial differential equation (NLPDE) needs first the study of its integrability, such as, the existence of a sufficiently large number of conservation laws or symmetries [1-4]. Many methods are used for studying the integrability of nonlinear partial differential equations. Among them the singular manifold method based on Painlevè analysis [5-7], homogeneous balance method [8-11], Weiss, Tabor and Carnevale (WTC) method [12], symbolic computation method [13] and Bäcklund transformation (BT) [14]. We here derive the Lax pair for Calogero-Bogoyavlenskii-Schiff (CBS) equation [15-19];

$$u_{xt} + u_x u_{xy} + \frac{1}{2} u_{xx} u_y + \frac{1}{4} u_{xxxy} = 0 \tag{1}$$

This equation describes the (2+1) dimensional interaction of Riemann wave propagating along the y-axis with long wave propagating along the x-axis [15-19]. CBS equation was investigated from various perspectives, such as the classical and non-classical methods. Through several symmetry reductions, exact solutions of the CBS equation were derived [20], while a variety of exact solutions using the improved (G`/G)-expansion method were presented [21-23], the symbolic computation method [24, 25], the exponential expansion method [26], the improved tanh-coth method [27], the symmetry method [28], the Hirota's bilinear method to derive its multiple front solutions [29]. Here the singular manifold method is used to deduce the CBS Lax pair. Then we proceed to a similarity reduction of this Lax pair to a system of ordinary differential equations obtain optimal similarity solutions and compare our results with previous work on CBS equation. The organization of this paper is as follows: In Section 2 the Lax pair is deduced for CBS equation. In Section

3 the similarity solutions for this Lax pair are deduced. Finally we present the conclusions in section 4.

2. Singular Manifolds Method

In this section, the Singular Manifold Method is applied to find the BT and Lax pair for the (2+1) dimensional CBS equation (1). Singular Manifold Method is an inverse solution [30-32] of nonlinear partial differential equations having a series form;

$$u(x,y,t) = \sum_{j=0}^{\infty} u_j(x,y,t)\phi(x,y,t)^{j-\alpha} \tag{2}$$

Where $\phi(x,y,t)$ is an Eigen function and α is a real number obtained from the dominant behavior analysis.

2.1. Bäcklund Transformation of CBS Equation

Replacing for (2) into (1), the dominant behavior analysis yields α=1, in this case the series expansion (2) reduces to:

$$u = u_0\,\varphi^{-1} + u_1 \tag{3}$$

This is the Bäcklund transformation of the Calogero-Bogoyavlenskii-Schiff equation. Substitute from (3) into (1), then equating the coefficients of the similar powers of φ to zero yields;

Coefficient of φ^{-4};

$$u_0 = 2\varphi_x \tag{4}$$

Replacing for u_0 in (3) reduces it to;

$$u = \frac{2\varphi_x}{\varphi} + u_1 = 2(ln\varphi)_x + u_1 \tag{5}$$

2.2. Lax Pair of CBS Equation

Equation (1) Lax pair's is deduced by substituting (5) into (1) and equating the coefficients of the similar powers of φ to zero giving;

Coefficient of φ^{-2}=0;

$$-4u_{1x}\varphi_{xy}\varphi_x - 3\varphi_x\varphi_{xx}u_{1y} - 2u_{1x}\varphi_{xx}\varphi_y - 2\varphi_x^2 u_{1xy} - 4\varphi_{xt}\varphi_x + \varphi_{xxy}\varphi_{xx} - 2\varphi_{xxxy}\varphi_x - u_{1xx}\varphi_x\varphi_y - 2\varphi_{xx}\varphi_t - \frac{1}{2}\varphi_{xxxx}\varphi_y = 0 \tag{6}$$

Coefficient of φ^{-3}=0;

$$4u_{1x}\varphi_x^2\varphi_y + 2\varphi_x^3 u_{1y} + 2\varphi_x^2\varphi_{xxy} + 4\varphi_x^2\varphi_t + 2\varphi_{xxx}\varphi_x\varphi_y - 2\varphi_x\varphi_{xx}\varphi_{xy} - \varphi_{xx}^2\varphi_y = 0 \tag{7}$$

Then defined new variables *V, R* and *Z* as follows;

$$\mathrm{V} = \frac{\varphi_{xx}}{\varphi_x}, \mathrm{R} = \frac{\varphi_t}{\varphi_x} \text{ and } \mathrm{Z} = \frac{\varphi_y}{\varphi_x} \tag{8}$$

Substitute (8) into (6) and (7) leads to the two equations;

$$-6u_{1x}ZV - 4u_{1X}Z_X - 3Vu_{1y} - 2u_{1xy} - u_{1xx}Z - \frac{7}{2}ZVV_x - 6RV - 4R_x + VV_y - V^2Z_x - \frac{3}{2}ZV^3 - 2V_{xy} - 4VV_y - 2Z_xV_x - \frac{1}{2}ZV_{xx} = 0 \tag{9}$$

$$-4R = 4u_{1x}Z + 2u_{1y} + 2V_y + 2V_xZ + V^2Z \tag{10}$$

Equations (9) and (10) can be easily linearized by introducing a new function ψ defined as:

$$\varphi_x = \psi^2 \tag{11}$$

By substituting (11) into (8) yields;

$$V = 2\frac{\psi_x}{\psi} \tag{12}$$

$$Z_x + ZV = 2\frac{\psi_y}{\psi} \tag{13}$$

$$R_x + RV = 2\frac{\psi_t}{\psi} \tag{14}$$

Then, by substituting (12), (13) and (14) into (9) and (10) respectively, we get:

$$\left(-4u_{1x}\psi_x - u_{1xx}\,\psi - \psi_{xxx} - 3\frac{\psi_x\psi_{xx}}{\psi}\right)Z - 8u_{1x}\psi_y - 6u_{1y}\psi_x - 2u_{1xy}\psi - 4R\psi_x - 8\psi_t - 4\psi_{xxy} - 4\frac{\psi_y\psi_{xx}}{\psi} + 4\psi_y\psi_x^2 = 0 \tag{15}$$

$$-4R = 4u_{1x}Z + 2u_{1y} + 4\frac{\psi_{xy}}{\psi} - 4\frac{\psi_x\psi_y}{\psi^2} + 4Z\frac{\psi_{xx}}{\psi} \tag{16}$$

By replacing for (16) into (15) provides us with two equations;

$$-u_{1xx}\,\psi - \psi_{xxx} + \frac{\psi_x\psi_{xx}}{\psi} = 0 \tag{17}$$

$$-8u_{1x}\psi_y - 4u_{1y}\psi_x - 2u_{1xy}\psi - 8\psi_t - 4\psi_{xxy} - 4\frac{\psi_y\psi_{xy}}{\psi} + \frac{4\psi_y\psi_{xx}}{\psi} = 0 \tag{18}$$

Dividing (17) by ψ and integrating with respect to 'x' leads to the first Calogero-Bogoyavlenskii Lax pair

$$\psi_{xx} + (u_{1x} - \lambda)\psi = 0 \tag{19}$$

Where λ is a constant of integration. Then setting ψ_x=ψ_y in (18) we obtain the second Lax pair

$$4\psi_t + 4u_{1x}\psi_y + 2u_{1y}\psi_x + u_{1xy}\psi + 2\psi_{xxy} = 0 \tag{20}$$

3. The Similarity Solutions of CBS Lax Pair

3.1. Lie Infinitesimals of CBS Equation

The Lie infinitesimals of the CBS Lax pair (19) and (20) have the form;

$$V_1 = f_1(t)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + [\,2y\, f_{1t}(t) + f_2(t)]\frac{\partial}{\partial u} \tag{21}$$

$$V_2 = f_3(t)\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + [\,2y\, f_{3t}(t) + f_4(t)]\frac{\partial}{\partial u} \tag{22}$$

$$V_3 = f_5(t)\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi} + [\,2y\, f_{5t}(t) + f_6(t)]\frac{\partial}{\partial u} \tag{23}$$

$$V_4 = \left[f_7(t) + \frac{x}{3}\right]\frac{\partial}{\partial x} + \frac{y}{3}\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + \left[\,2y\, f_{7t}(t) + \frac{\lambda}{3}(2x - 4y) - \frac{u}{3} + f_8(t)\right]\frac{\partial}{\partial u} \tag{24}$$

The arbitrary functions f_i *(t)*, *i=1.8*, are optimized through the commutative products listed in Table 1. This leads to a system of ordinary differential equations in the unknown functions f_i *(t)* reported here;

Table 1. *Commutator table.*

	V$_1$	V$_2$	V$_3$	V$_4$
V$_1$	0	0	0	V$_1$
V$_2$	0	0	0	$\frac{V_2}{3}$
V$_3$	0	0	0	0
V$_4$	-V$_1$	$\frac{-V_2}{3}$	0	0

$$\begin{cases} f_{7t} = \frac{2}{3}f_1 + tf_{1t}, f_{4t} - 2f_{1t} = 0 \\ -\frac{2}{3}f_4 - tf_{4t} = \frac{4\lambda}{3} - 2f_{7t}, \\ 2yf_{7tt} + f_{8t} = 2tf_{1tt}y + \frac{10}{3}f_{1t}y + \frac{4}{3}f_2 + tf_{2t} - \frac{2\lambda}{3}f_1 \\ f_{3t} = 0, f_{5t} = 0, f_{6t} = 0 \end{cases} \tag{25}$$

Solving this system of ODE's (25), leads to the values of functions f_i *(t)*, *i=1…8*, listed below;

$$f_1(\mathrm{t}) = \frac{\lambda}{3}, f_2(\mathrm{t}) = 1, f_3(\mathrm{t}) = 0, f_4(\mathrm{t}) = \frac{-4\,\lambda}{3}, f_5(\mathrm{t}) = 0,$$

$$f_6(\mathrm{t}) = 0, f_7(\mathrm{t}) = \frac{2\lambda}{9}\,\mathrm{t} \text{ and } f_8(\mathrm{t}) = \left(\frac{4}{3} - \frac{2\lambda^2}{9}\right)\mathrm{t} \tag{26}$$

According to these values the Lie vectors (21) to (24) is rewritten as:

$$V_1 = \frac{\lambda}{3}\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{\partial}{\partial u} \tag{27}$$

$$V_2 = \frac{\partial}{\partial y} + \left[\frac{-4\,\lambda}{3}\right]\frac{\partial}{\partial u} \tag{28}$$

$$V_3 = \psi\frac{\partial}{\partial \psi} \tag{29}$$

$$V_4 = \left[\frac{2\lambda}{9}\,\mathrm{t} + \frac{x}{3}\right]\frac{\partial}{\partial x} + \frac{y}{3}\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + \left[\frac{4\lambda}{9}y + \frac{\lambda}{3}(2x - 4y) - \frac{u}{3} + \left(\frac{4}{3} - \frac{2\lambda^2}{9}\right)\mathrm{t}\right]\frac{\partial}{\partial u} \tag{30}$$

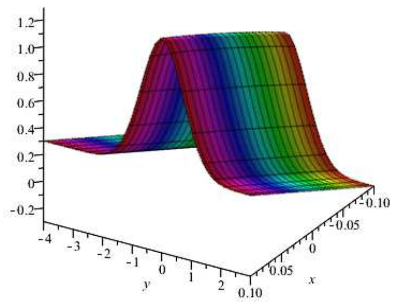
Vectors V$_1$ to V$_4$ are used to reduce and solve the Lax system (19) and (20).

Vector V$_1$ is used to reduce the system of equations (19) and (20), then solve it giving the following two solutions;

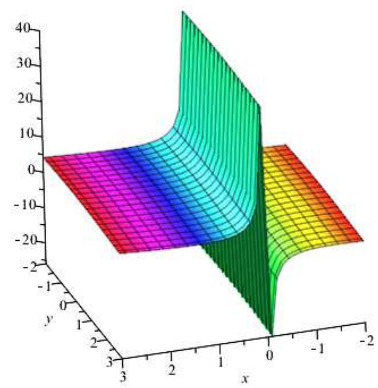
$$u_1 = F_1(y) + t + \lambda x - \frac{1}{3}\lambda^2 t \tag{31}$$

$$u_2 = \frac{3x + F_2\left(y, \frac{-3x + \lambda t}{\lambda}\right)\lambda}{\lambda} \tag{32}$$

Where F_1 is an arbitrary function of y, F_2 is an arbitrary function of (y, t) and λ is a constant of integration. These solutions are plotted for=1, $F_1 = \frac{\sin y}{y}$, $F_2 = \frac{e^{-y^2}\sin(t-3x)}{(t-3x)}$, in (Figure 1(a, b)) for *t=0.1* and in (Figure 2 (a, b)), for t=1.

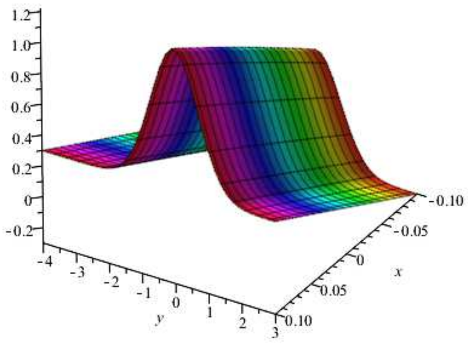


(a) $u_1(x, y, t) = 3x + \frac{e^{-y^2}\sin(0.1 - 3x)}{(0.1 - 3x)}$

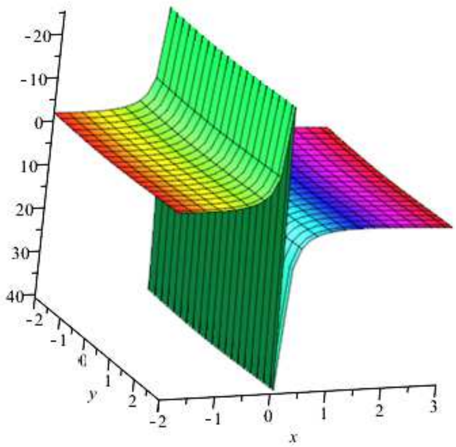


(b) $u_2(x, y, t) = \frac{2}{x} + \frac{\sin y}{y} + x + \frac{1}{15}$

Figure 1. *Solutions of CBS equation for vector V_1 at time t=0.1 and t=1.*



$(a)\ u_3(x,y,t) = 3x + \dfrac{e^{-y^2}\sin(1-3x)}{(1-3x)}$



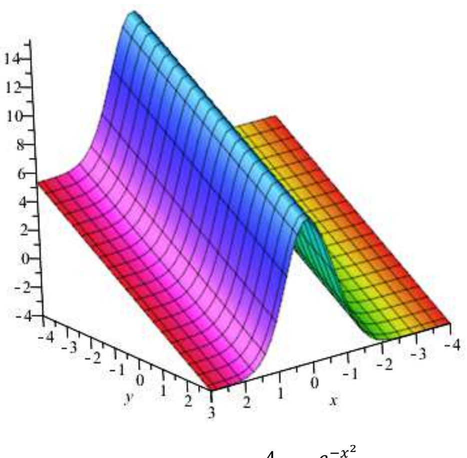
(b) $u_4(x,y,t) = \dfrac{2}{x} + \dfrac{siny}{y} + x + \dfrac{2}{3}$

Figure 2. *Solutions of CBS equation for vector V_1 at time t=0.1 and t=1.*

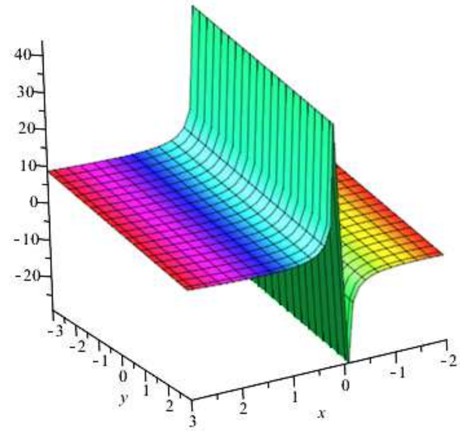
The solution of the Lax system (19) - (20) by using the vector V_2 are;

$$u_3 = \frac{-4}{3}\lambda y + F_3(x,t) \quad (33)$$

$$u_4 = F_4(t) + \frac{1}{3}(-4y+3x)\lambda \quad (34)$$



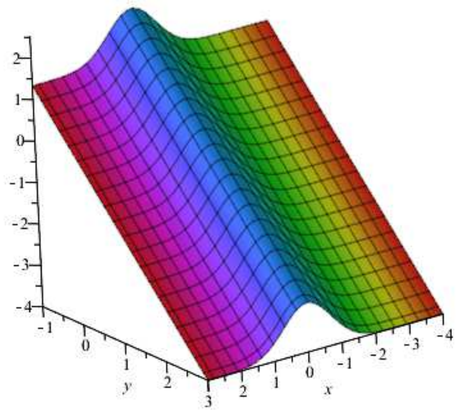
$(a) u_5(x,y,t) = -\dfrac{4}{3}y + \dfrac{e^{-x^2}}{0.1}$



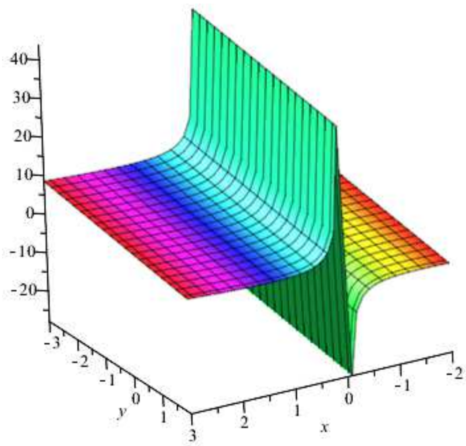
$(b)\ u_6(x,y,t) = \dfrac{2}{x} + \dfrac{\sin(0.1)}{0.1} + \dfrac{1}{3}(-4y+3x)$

Figure 3. *Solutions of CBS equation for vector V_2 for t=0.1 and t=1.*

Where F_3 is an arbitrary function of (x, t) and F_4 is an arbitrary function of (t). Choosing, $F_3 = \frac{e^{-x^2}}{t}, F_4 = \frac{\sin t}{t}$, the solutions (33) and (34) are plotted for $\lambda = 1$ as depicted in (Figure 3(a, b)), for *t=0.1* and in (Figure 4(a, b)), for *t=1*.



$(a)\ u_7(x,y,t) = -\dfrac{4}{3}y + \dfrac{e^{-x^2}}{1}$



(b) $u_8(x,y,t) = \dfrac{2}{x} + \dfrac{\sin(1)}{1} + \dfrac{1}{3}(-4y+3x)$

Figure 4. *Solutions of CBS equation for vector V_2 for t=0.1 and t=1.*

3.2. Comparison with Previous Works

We do then compare results obtained using vectors V_1 and V_2, (31)-(34) with previous solutions of (2+1)-dimensional CBS equation as in the following.

Bruzon and Gandarias [20] used the classical and non-classical symmetry methods to obtain symmetry reductions and exact solutions of the (2+1)-dimensional integrable Calogero–Bogoyavlenskii–Schiff equation. They obtained the solution of (2+1)-dimensional CBS equation;

$$u(x,y,t) = 3k\,sech^2\left(\sqrt{\frac{3k}{2}}\left(x + a(y,t)\right)\right) + k_1 \qquad (35)$$

Where $a = a(y,t)$, satisfies $a_t = \emptyset(a)a_z$ and k_1 is an arbitrary constant. Here the similarity variable $x + a(y,t)$ connects y to t, while in our results in (31) and (32), contains the d'Alembert form for x, t and arbitrary functions giving many soliton shapes. Variety of exact solutions [21-24] of Calogero–Bogoyavlenskii–Schiff equation are constructed by using the improved (G'/G) expansion method. Family of exact solutions of CBS equation are obtained. The exact solution take the solitary wave form [21], when $A < 0$

$$u(\xi) = \alpha_0 + 2\sqrt{-A}\tanh\left(\sqrt{-A}\xi\right) = \alpha_0 + 2\sqrt{-A}\coth(\sqrt{-A}\xi) \qquad (36)$$

Where $\xi = x + z + 4At$, α_0 and A are arbitrary constants The exact solution is for $\Delta_1 = \frac{\sqrt{\lambda^2-4\mu}}{2}$ [22];

$$u_1 = -\frac{\left(-4k^2\varphi_y\mu + k^2\varphi_y\lambda^2 + \tau_t\right)x}{4\varphi_y} + 2k\left(-\frac{\lambda}{2} + \Delta_1\left(\frac{C_1\cosh(\Delta_1(kx+\varphi+\tau)) + C_2 sinh(\Delta_1(kx+\varphi+\tau))}{C_1 sinh(\Delta_1(kx+\varphi+\tau)) + C_2\cosh(\Delta_1(kx+\varphi+\tau))}\right)\right) \qquad (37)$$

The (*G'/G*) expansion method was used for $\lambda^2 - 4\mu >$ [23] yields;

$$u(\xi) = \pm\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A_1 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right) \qquad (38)$$

$$v(\xi) = \frac{\lambda^2-4\mu}{8}\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A_1 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)^2 \qquad (39)$$

The solution takes the form when $\lambda^2 - 4\mu > 0$ [24]

$$u_1 = -\frac{3}{2}f'^2(y)\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)^2 + \frac{3}{2}\left(f''(y) - f'^2(y)\lambda\right)\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right) + \frac{1}{4}\left(-2\mu f'^2(y) - \lambda^2 f'^2(y) + 3\lambda f''(y) - f'''(y)/f'(y)\right) \qquad (40)$$

$$v_1 = -\frac{3}{2}kf'(y)\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)^2 - \frac{3}{2}k\lambda f'(y)\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1 cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right) - \frac{1}{4}\left(6\mu kf'(y) + g'(t)\right) \qquad (41)$$

where $\xi = k(x) + f(y) + g(t)$, C_1 and C_2 are arbitrary constants. Biao and Yong [25] obtained some exact analytical solutions, which contain soliton and periodic solutions to the generalized Calogero–Bogoyavlenskii–Schiff (GCBS) equation by using generalized Riccati equation expansion method and symbolic computations. They get the exact analytical solution of the GCBS equation;

$$u(x,y,t) = 2\sqrt{\frac{-y}{R_t}}R + F_2(t) \mp \sqrt{\frac{y}{t}}\tanh\{\sqrt{-R} \pm \left[\sqrt{-\frac{y}{4R_t}}x - ln\left(C_1\sqrt{\frac{y}{t}} + C_2\right) - (1+\alpha)\,ln(y) + \alpha\,ln(t)\right] \qquad (42)$$

Alam and Tunc [26] applied the exponential function expansion method to construct exact solutions of the nonlinear Bogoyavlenskii equation. The solution was found when $\mu \neq 0$, $\lambda^2 - 4\mu > 0$ is

$$u(x,y,t) = \frac{1}{2}\lambda - \left(\frac{2\mu}{\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+E)\right)+\lambda}\right) \qquad (43)$$

Where $\xi = x - \left(-\frac{1}{8}\lambda^2 + \frac{1}{2}\mu\right)t$ and E is an arbitrary constant. Cesar and Gomez [27] used an improved tanh-coth method to obtain exact solutions of the Bogoyavlenskii equation. The exact solutions of the Bogoyavlenskii equation are

$$u(x,y,t) = \frac{1}{x+y+\frac{1}{32}(48a_1a_2-32k_1)t+\varepsilon_0} + \frac{a_1a_2(x+y+\frac{1}{32}(48a_1a_2-32k_1)t+\varepsilon_0)}{1-\sqrt{a_1a_2}(x+y+\frac{1}{32}(48a_1a_2-32k_1)t+\varepsilon_0)} \qquad (44)$$

$$v(x,y,t) = \frac{u^2(x,y,t)}{2} - k_1 \qquad (45)$$

with a_1, a_2, k_1 and ε_0 arbitrary constants. The symmetry method has been carried over [28] to the Calogero Bogoyavlenskii Schiff equation to find exact solutions of this equation. The exact solution appears as the following

$$u(x,y,t)=\frac{1}{2}yf(t)+A_0+\frac{2}{x-\int f(t)dt+y} \quad (46)$$

Wazwaz [29] employed the Hirota's bilinear method to derive multiple-front solutions for the Calogero–Bogoyavlenskii–Schiff equation. He obtained the solution of the CBS equation as following:

$$u(x,y,t)=\frac{2k_1e^{k_1x+m_1y-k_1^2m_1t}}{1+e^{k_1x+m_1y-k_1^2m_1t}} \quad (47)$$

Saleh *et al.* [33] obtained exact solutions of Calgero–Bogoyavlenskii– Schiff equation by using the singular manifold method after Lie reductions. They obtained the exact solutions of Calgero–Bogoyavlenskii– Schiff equation as:

$$u(x,y,t)=\frac{2c_3(\sec\frac{\sqrt{c_1}}{2}(x-2\sqrt{y}-2\sqrt{t}+c_2))^2}{\frac{2c_3}{\sqrt{c_1}}\tan\left(\frac{\sqrt{c_1}}{2}(x-2\sqrt{y}-2\sqrt{t}+c_2)\right)+c_4}-\sqrt{c_1}\tan\frac{\sqrt{c_1}}{2}\left(x-2\sqrt{y}-2\sqrt{t}+c_2\right)+\frac{c_1}{6}\left(x-2\sqrt{y}-2\sqrt{t}\right)+\frac{c_1c_2}{6}+\frac{2(y-1)}{\sqrt{t}} \quad (48)$$

where, $c_1=4$, $c_3=1$ and $c_2=c_4=c_5=0$. Kumar [34] used the similarity transformations method via Lie-group theory to derive exact solutions of (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation. The result obtained shows a linear of x, y terms weighted by t^{ε}, where ε=1-a, a-1 or t-1. The solution of CBS equation is

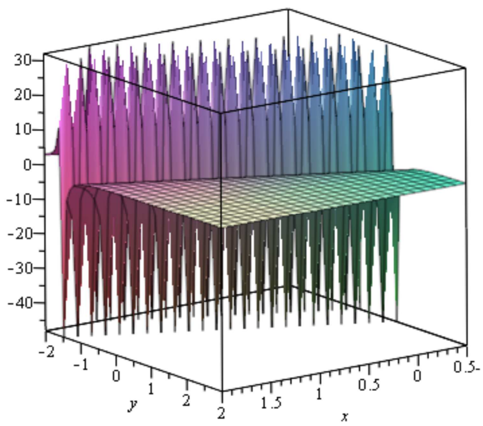
$$u(x,y,t)=\frac{t^{(t-1)/2}}{C_2}+\frac{C}{t^{(1-a)/2}}+\frac{B(1-a)}{t^{(1+a)/2}}y\left(xt^{(a-1)/2}-2A\frac{t^{(a-1)/2}}{C_2(a-1)}-4B+B_1\right)+\frac{y}{4t}+\int\frac{F_2(t)}{t^{(a+1)/2}}dt \quad (49)$$

Gandarias and Bruzon [35] obtained the solution of the (2+1)-dimensional integrable CBS equation by using classical Lie symmetries and travelling-wave reductions with variable velocity depending on the form of an arbitrary function. The solution of (CBS) equation

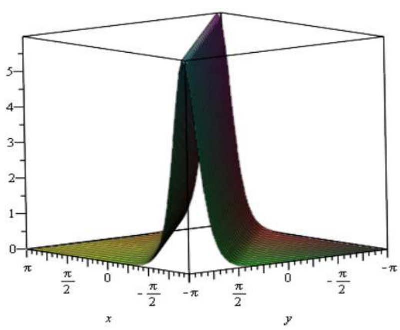
$$u(x,y,t)=\sqrt{2}\tanh\left(\frac{x-f(y-\lambda t)}{\sqrt{2}}\right) \quad (50)$$

where $\lambda=\frac{1}{2}$, $f(y-\lambda t)=y-\frac{t}{2}$, t=0.1.

Some of the previous obtained results are hereafter plotted.

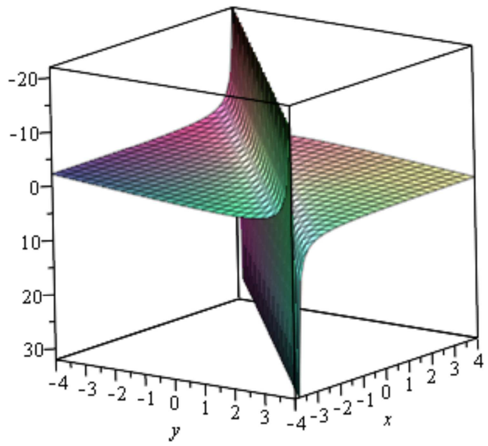


$(a)\ u=6sech^2\left(\sqrt{3}(x+y-t)\right)$

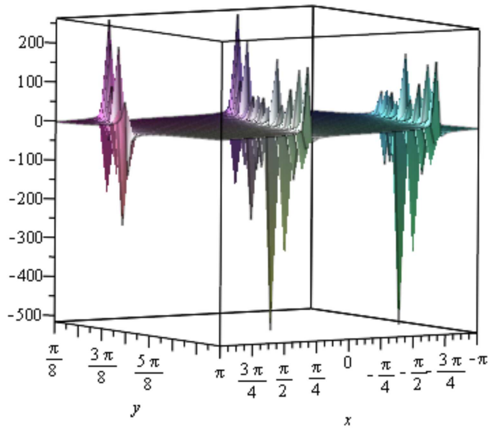


$$(b)u=\frac{1}{x+y+t}+\frac{x+y+t}{1-(x+y+t)}$$

Figure 5. *The soliton solution of Bruzon and Gandarias [20] and periodic solution of Cesar and Gomez [27].*

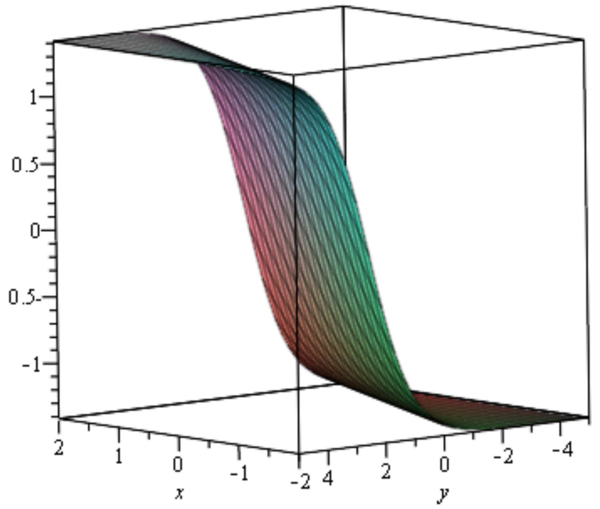


$$(a)\ u=\frac{y}{2}+\frac{2}{x+y-t}$$



$$(b)\ u=\frac{2(sec(x-2\sqrt{y}-2\sqrt{t}))^2}{\tan\left((x-2\sqrt{y}-2\sqrt{t})\right)}-2tan(x-2\sqrt{y}-2\sqrt{t})+\frac{4}{6}(x-2\sqrt{y}-2\sqrt{t})+\frac{2(y-1)}{\sqrt{t}}$$

Figure 6. *CBS solutions of Moatimid et al. [28] and Saleh et al. [33] respectively.*



$$u = \sqrt{2}\tanh\left(\frac{x+y-t}{\sqrt{2}}\right)$$

Figure 7. *Solution of Gandarias and Bruzon [35].*

It's clear from this comparison that we derive a new solution of Calogero–Bogoyavlenskii-Schiff equation by using a new method different from the previous findings.

4. Conclusion

Lax pair of (2+1) Calogero-Bogoyavlenskii-Schiff equation is obtained by using the singular manifold method. The detected Lie infinitesimals for the CBS Lax pair's contains eight unknown functions that are specialized by the aide of the commutator table. These functions are evaluated through the solution of a set of linear differential equations. Their solutions lead to optimal Lie vectors. The CBS Lax pair is reduced by using the optimal Lie vectors to a system of ODEs. New solutions for CBS equation are obtained and plotted for different arbitrary functions, reveal some solitary waves in the form of soliton and kink waves. The obtained solutions are compared with previous works. The comparison reveals that, the derived solutions are new and the detection of the Lax pair solution's is effective in exposure traveling wave solutions of nonlinear evolution equations.

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