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Advance Exp (-Φ(ξ)) Expansion Method and Its Application to Find the Exact Solutions for Some Important Coupled Nonlinear Physical Models

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Abstract: The theoretical investigations of resonance physical phenomena by nonlinear coupled evolution equations are become important in currently. Hence, the purpose of this paper is to represent an advance exp (-Φ(ξ))-expansion method with nonlinear ordinary differential equation for finding exact solutions of some nonlinear coupled physical models. The present method is capable of evaluating all branches of solutions simultaneously and this difficult to distinguish with numerical technique. To verify its computational efficiency, the coupled classical Boussineq equation and (2+1)-dimensional Boussinesq and Kadomtsev-Petviashili equation are considered. The obtained solutions in this paper reveal that the method is a very effective and easily applicable of formulating the exact traveling wave solutions of the nonlinear coupled evolution equations arising in mathematical physics and engineering.

Keywords: Coupled Classical Boussinesq Equation, Boussinesq-Kadomtsev-Petviashili Equation, Solitary Wave Solution, Periodic Wave Solution

1. Introduction

The coupled nonlinear evolution equations (NLEEs) are widely used to describe many physical mechanisms of natural phenomena and dynamical processes in mathematical physics and engineering. The investigation of exact solutions of NLEEs plays an important role of these intricate physical phenomena and would be helpful to observe the wave translations in many branches of physics, chemistry and biology. In this article, we highlighted an analytical method, namely advance exp(-Φ(ξ)) -expansion method for finding more valuable explicit solutions of NLEEs. The valuable explicit form solutions provide a means to describe the salient features in various science, technology and engineering applications. It can be serve as a basis for perfecting and testing computer algebraic software, such as Maple, Mathematica, MatLab etc for solving NLEEs. It is notable that many nonlinear partial differential equations (NPDEs) of physics, chemistry and biology contain unknown parameters and unknown functions. Exact solutions permit research scholars to design and run experiments, by creating appropriate natural conditions, to determine these functions and parameters. There are several types of well-known methods that have been developed to construct analytical solutions of NLEEs such as the (G'/G)-expansion method [1-5], the modified simple equation method [6, 7], the tanh method [9-10], the Homotopy perturbation technique [11], the homogeneous balance method [12-14], the Hirota method [15], the Kudryashov method [16, 17], the Exp-function method [18, 19], the improved F-expansion method [20], the exp $(-\varphi(\xi))$ -expansion method [21-25] and so on. The choice of an appropriate ansatz is of great importance when using these analytical methods. Among these approaches, the proposed advance exp(-Φ(ξ)) -expansion method is easily applicable with the help of symbolic computation and powerful mathematical method to obtain more general solitary and periodic wave solutions of NLEEs in mathematical physics and engineering. The main idea of this technique is to express the exact traveling wave solutions of NLEEs in terms of trigonometric, hyperbolic and rational

functions that satisfy the nonlinear ordinary differential equation (ODE) $\Phi'(\xi)+\lambda\exp(\Phi(\xi))+\mu\exp(-\Phi(\xi))=0$, where λ and μ real parameters. The advantage of the proposed method over the other existing methods is that it provides some new exact traveling wave solutions to the nonlinear PDEs. Algebraic manipulations of this method is also much easier rather than the others existing methods.

There are several types of coupled NLEEs that appeared as model equations in mathematical physics, chemistry and biology, such as the coupled classical Boussineq equation appeared as a model equation to describe dispersive long gravity wave traveling in two horizontal directions on shallow water of uniform depth, and the (2+1) dimensional Boussinesq and Kadomtsev-Petviashili (BKP) equation also appeared as a model equation to describe various types of wave phenomena in mathematical physics and so on. The dispersive and nonlinear effects of these coupled NLEEs may be balanced and thus the solitary wave causes. The existence and appearance of solitary waves in complicated physical problems apart from the model equations of mathematical physics must be analyzed with sufficient accuracy. Therefore, the aim of this article is to explore a new study linking to the advance exp(-$\Phi(\xi)$) -expansion method for solving the coupled NLEEs to demonstrate the effectiveness and truthfulness of this method.

The rest of the paper has been prepared as follows. In section 2, the proposed advance exp(-$\Phi(\xi)$) -expansion method is discussed in details. The section 3 presents the application of this method and physical explanation of the determined solutions. The advantages of this method and comparison with other methods are presented in section 4. Conclusions have been drawn in Section 5.

2. The Methodology

This section presents the briefly descriptions of the proposed advance exp $(-\Phi(\xi))$ -expansion method.

Let us consider general NLEEs as

$$f(u,u_t,u_x,u_{xx},u_{tt},u_{tx},\ldots)=0\,, \tag{1}$$

where, $u(x,t)$ is an empirical function, f is a polynomial in $u(x,t)$ and its partial derivatives in which the higher order derivatives and nonlinear terms are involved.

The exp (-Φ(ξ))-expansion method [21-25] has been employed to look into exact solutions, wherein the nonlinear ODE $\Phi'(\xi)=\exp(-\Phi(\xi))+\mu\exp(\Phi(\xi))+\lambda$; $\lambda,\mu\in\Re$ provides only a few traveling wave solutions to the nonlinear NLEEs. From the references [21-25], we observe that the method does not give rich solutions to comprehend inner picture to the NLEEs. Therefore, in order to get more traveling wave solutions and to understand the inner structure apparently of the nonlinear physical phenomena, in this article we choose the ODE $\Phi'(\xi)+\lambda\exp(\Phi(\xi))+\mu\exp(-\Phi(\xi))=0$ as auxiliary equation and the general solutions of this ODE have been used. The descriptions of this proposed method as follows:

If we combine the compound variables x and t by a single variable, that is, $u(x,t)=u(\xi)$, $\xi=k\,x\pm V\,t$, where k and V arbitrary constants, then the equation (1) can be reduce to an ODE as

$$F(u,u',u'',u''',\cdots\cdots)=0\,, \tag{2}$$

where, F is a function of $u,u',u'',u''',\cdots\cdots$ and primes indicate the ordinary derivatives with respect to ξ.

The proposed method allow us to write the traveling wave solution to the equation (2) in the following form

$$u(\xi)=\sum_{i=0}^{N}a_i\left(e^{-\Phi(\xi)}\right)^i,\ a_N\neq 0 \tag{3}$$

where the coefficients $a_i\,(0\le i\le N)$ are constants to be determined and $\Phi=\Phi(\xi)$ satisfies the first order nonlinear ODE:

$$\Phi'(\xi)+\lambda\exp(\Phi(\xi))+\mu\exp(-\Phi(\xi))=0,\lambda,\mu\in\Re. \tag{4}$$

It is notable that eq. (4) has the following six kinds of general solutions as follows:

$$\Phi(\xi)=-\ln\left(\sqrt{\frac{\lambda}{\mu}}\tan\left[\sqrt{\lambda\mu}\ (\xi+\xi_0)\right]\right),\lambda\mu>0,$$

$$\Phi(\xi)=-\ln\left(-\sqrt{\frac{\lambda}{\mu}}\cot\left[\sqrt{\lambda\mu}\ (\xi+\xi_0)\right]\right),\lambda\mu>0,$$

$$\Phi(\xi)=-\ln\left(\sqrt{\frac{\lambda}{-\mu}}\tanh\left[\sqrt{-\lambda\mu}\ (\xi+\xi_0)\right]\right),\lambda\mu<0,$$

$$\Phi(\xi)=-\ln\left(\sqrt{\frac{\lambda}{-\mu}}\coth\left[\sqrt{-\lambda\mu}\ (\xi+\xi_0)\right]\right),\lambda\mu<0, \tag{5}$$

$$\Phi(\xi)=-\ln\left(-\frac{1}{\mu(\xi+\xi_0)}\right),\lambda=0,\ \mu>0,$$

$$\Phi(\xi)=-\ln\left(\lambda(\xi+\xi_0)\right),\lambda\in\Re,\ \mu=0,$$

where ξ_0 is the integrating constant and $\lambda\mu>0$ or $\lambda\mu<0$ depends on sign of λ. We have been cheeked the solutions (5) by putting back into the original equation (4) found correct.

The value of the positive integer N can be determined by balancing the higher order derivative term and the nonlinear term appearing in ODE (2). If the degree of $u=u(\xi)$ Is $D[u(\xi)]=n$, then the degree of the other expressions will be found by the following formulae:

$$D[\frac{d^pu(\xi)}{d\xi^p}]=n+p,\quad D[u^p\left(\frac{d^qu(\xi)}{d\xi^q}\right)^s]=np+s(n+q).$$

By substituting (3) into (2) together with the value of N, we obtain polynomials in $e^{-\Phi(\xi)}$. We set each coefficients of the resulting polynomial to zero, yielding a system of algebraic equations for $a_i\,(0 \le i \le N), k, \lambda, \mu, V$. With the help of symbolic computation, such as Maple, we can evaluate the obtaining system and find out the values $a_i\,(0 \le i \le N), k, \lambda, \mu, V$. Therefore, we are obtained the new multiple explicit solutions of NLEEs (1) by combining the equations (3) and (5) and inserting the value of $a_i\,(0 \le i \le N), k, \lambda, \mu$ and V.

3. Applications to Some Important Nonlinear Coupled Physical Models

The section presents the application of the advance exp(-Φ(ξ)) -expansion method to find the exact traveling wave solutions of the coupled nonlinear evolution equations and physical explanations of the determined solutions.

3.1. The Coupled Classical Boussineq Equation

This subsection present the new study linking to the famous coupled (1+1)-dimensional classical Boussineq equation through the advance exp(-Φ(ξ)) -expansion method.

The (1+1)-dimensional classical Boussineq equation is given as follows:

$$\left.\begin{aligned} &v_t + \{(1+v)u\}_x + \frac{1}{3}u_{xxx} = 0 \\ &u_t + uu_x + v_x = 0 \end{aligned}\right\}. \tag{6}$$

Here $u(x,t)$ and $v(x,t)$ represents the evolutions the surface in any natural varied instances. The coupled equation is an important class of NPDEs, which was first introduced by Wu and Zhang [26] for modeling nonlinear and dispersive long gravity wave traveling in two horizontal directions on shallow water of uniform depth. There is an amount of literature [24, 26-28], where the coupled equation is well studied. Gepreel [27] have been studied some traveling wave solutions via the generalized (G'/G)-expansion method. Zyed and Joudi [28] has found more traveling wave solution through the extended (G'/G)-expansion method. Recently, Roshid and Rahman [24] also studied the periodic and solitons solutions of the equations (6) through the exp(-Φ(ξ))-expansion method. The solutions which are found in [24] does not give rich solutions to comprehend inner picture to these equations. In addition to get the more valuable explicit form solutions to the classical Boussineq equations, we introduce the compound variables x and t to a single variable as

$$u(x,t) = U(\xi),\ v(x,t) = w(\xi), \tag{7}$$

where $\xi = x - Vt$, the coupled equation (6) reduces to the ODE as

$$\left.\begin{aligned} &-Vw' + [(1+w)U]' + \frac{1}{3}U''' = 0 \\ &-VU' + UU' + w' = 0 \end{aligned}\right\} \tag{8}$$

Integrating (8) once with regard to ξ, we obtain

$$\left.\begin{aligned} &C - Vw + (1+w)U + \frac{1}{3}U'' = 0 \\ &K - VU + \frac{U^2}{2} + w = 0 \end{aligned}\right\} \tag{9}$$

where, C and K are integrating constants.

According to the proposed advance exp(-Φ(ξ)) -expansion method, the solutions of the equations (9) can be written as

$$U(\xi) = a_0 + a_1 e^{-\Phi(\xi)},\ a_1 \neq 0 \tag{10}$$

$$w(\xi) = b_0 + b_1 e^{-\Phi(\xi)} + b_2 \left(e^{-\Phi(\xi)}\right)^2,\ a_2 \neq 0 \tag{11}$$

By substituting (10) and (11) into the eq. (9) and equating the coefficients of $(e^{-\Phi(\xi)})^i,\ (i = 0,1,2,3,4,\cdots)$ equal to zero, we obtain a system of algebraic equations (for simplicity the algebraic equations are not displayed here). Solving the obtained system by using Maple, the following sets of solutions are obtained:

$$\left\{C = -\frac{1}{2}a_0^2 + \frac{2}{3}\lambda\mu + 1, K = -a_0, V = a_0,\ a_0 = a_0,\ a_1 = \frac{2}{\sqrt{3}}\mu, b_0 = -\frac{2}{3}\lambda\mu - 1, b_1 = 0, b_2 = -\frac{2}{3}\mu^2\right\} \tag{12}$$

By combining the equations (5), (10), (11) and (12), the Boussinesq equation (6) has the following new explicit solutions:

$$\left.\begin{aligned} u_1(x,t) &= \left\{a_0 + \frac{2}{\sqrt{3}}\sqrt{\lambda\mu}\tan\left[\sqrt{\lambda\mu}\,(x - a_0 t + \xi_0)\right]\right\},\ \lambda\mu > 0 \\ v_1(x,t) &= -\frac{2}{3}\lambda\mu - 1 - \frac{2}{3}\lambda\mu\tan^2\left[\sqrt{\lambda\mu}\,(x - a_0 t + \xi_0)\right], \lambda\mu > 0 \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} u_2(x,t) &= \left\{a_0 - \frac{2}{\sqrt{3}}\sqrt{\lambda\mu}\cot\left[\sqrt{\lambda\mu}\,(x - a_0 t + \xi_0)\right]\right\},\ \lambda\mu > 0 \\ v_2(x,t) &= -\frac{2}{3}\lambda\mu - 1 - \frac{2}{3}\lambda\mu\cot^2\left[\sqrt{\lambda\mu}\,(x - a_0 t + \xi_0)\right], \lambda\mu > 0 \end{aligned}\right\}, \tag{14}$$

$$\left.\begin{aligned} u_3(x,t) &= \left\{ a_0 + \frac{2}{\sqrt{3}}\sqrt{-\lambda\mu}\tanh\left[\sqrt{-\lambda\mu}\,\left(x - a_0 t + \xi_0\right)\right]\right\},\ \lambda\mu < 0 \\ v_3(x,t) &= -\frac{2}{3}\lambda\mu - 1 + \frac{2}{3}\lambda\mu\tanh^2\left[\sqrt{-\lambda\mu}\,\left(x - a_0 t + \xi_0\right)\right], \lambda\mu < 0 \end{aligned}\right\}, \tag{15}$$

$$\left.\begin{aligned} u_4(x,t) &= \left\{ a_0 + \frac{2}{\sqrt{3}}\sqrt{-\lambda\mu}\coth\left[\sqrt{-\lambda\mu}\,\left(x - a_0 t + \xi_0\right)\right]\right\},\ \lambda\mu < 0 \\ v_4(x,t) &= -\frac{2}{3}\lambda\mu - 1 + \frac{2}{3}\lambda\mu\coth^2\left[\sqrt{-\lambda\mu}\,\left(x - a_0 t + \xi_0\right)\right], \lambda\mu < 0 \end{aligned}\right\}, \tag{16}$$

$$\left.\begin{aligned} u_5(x,t) &= \left\{ a_0 - \frac{2}{\sqrt{3}}\frac{1}{(x - a_0 t + \xi_0)}\right\}, \lambda = 0,\ \mu > 0 \\ v_5(x,t) &= -\frac{2}{3}\lambda\mu - 1 - \frac{2}{3}\left\{\frac{1}{(x - a_0 t + \xi_0)}\right\}^2, \lambda = 0,\ \mu > 0 \end{aligned}\right\}. \tag{17}$$

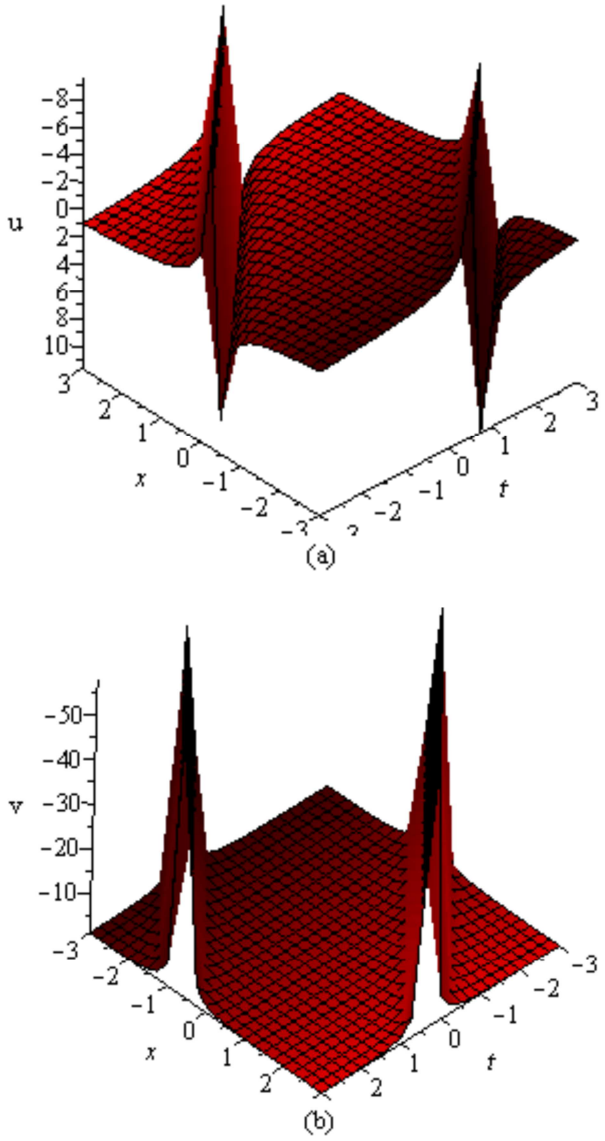


Figure 1. *Exact periodic traveling wave solutions of (a)* $u_1(x,t)$ *and (b)* $v_1(x,t)$ *with* $-3 \le x,t \le 3$.

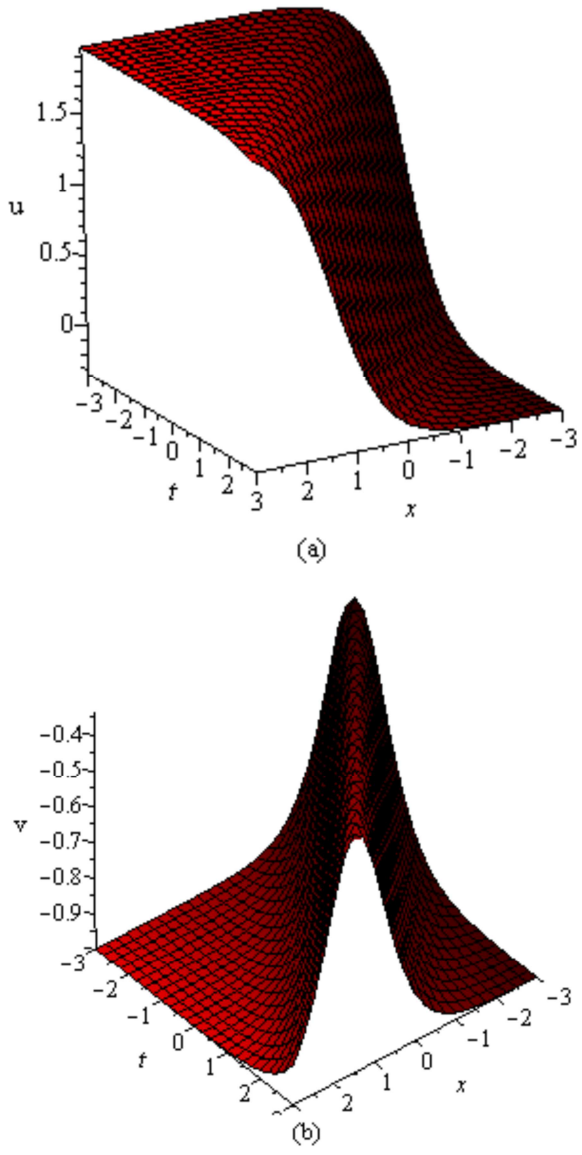


Figure 2. *Exact traveling wave solutions of (a) topological kink nature for* $u_3(x,t)$ *with* $-3 \le x,t \le 3$ *and (b) non-topological bell nature for* $v_3(x,t)$ *with* $-3 \le x,t \le 3$.

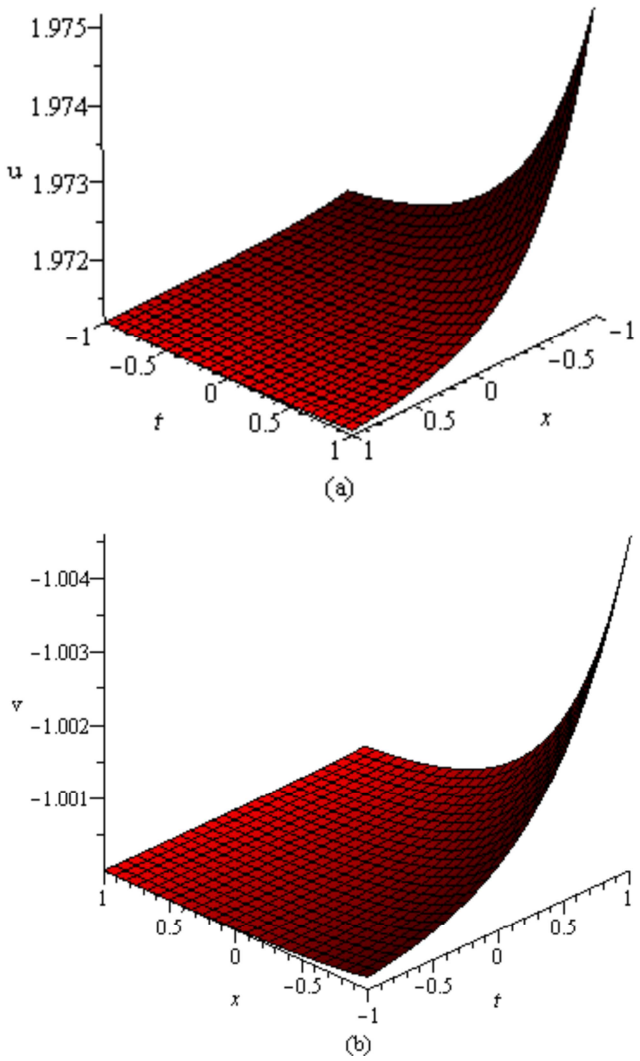


Figure 3. *Exact solitary wave solutions of 1-soliton nature for (a)* $u_4(x,t)$ *and (b)* $v_4(x,t)$ *with* $-1 \le x,t \le 1$.

From the above determined solutions, we observe that the proposed method according to auxiliary equation (4) gives more valuable explicit form solutions to the classical Boussinesq equation. The solutions of the classical Boussinesq equation are also represented various types of solitary and periodic wave solutions according to the variation of the physical parameters. Solutions $u_1(x,t)$ and $v_1(x,t)$ are presents the periodic traveling wave solutions with fixed parametric values $\lambda = 0.5, \mu = 0.5, a_0 = 1$ and $\xi_0 = 0.5$, which are shown in figure 1. Solutions $u_2(x,t)$ and $v_2(x,t)$ are also presents the periodic and singular type periodic wave solutions according to the fixed values $\lambda = 0.5, \mu = 0.5, a_0 = 1$ and $\xi_0 = 0.5$. Solution $u_3(x,t)$ represent the solitary wave solution of topological kink nature while $v_3(x,t)$ represent the solitary wave solution of non-topological bell nature corresponding to the values $\lambda = -1$, $\mu = 1$, $a_0 = \sqrt{2/3}$ and $\xi_0 = 1$, which are shown in figure 2. Solutions $u_4(x,t)$ and $v_4(x,t)$ are presents the solitary wave solutions of 1-soliton type with fixed parametric values $\lambda = -1$, $\mu = 1$, $a_0 = \sqrt{2/3}$ and $\xi_0 = 5$, which are shown in figure 3. Finally, solutions $u_5(x,t)$ and $v_5(x,t)$ are also represented the solitary wave solutions of soliton nature according to the variation of the parameters. These types of traveling wave solutions would be helpful for researcher to describe the dispersive long gravity wave propagation in two horizontal directions on shallow water of uniform depth or any natural varied physical instances. Other figures are ignored for convenience.

3.2. (2+1) Dimensional Boussinesq and Kadomtsev-Petviashili (BKP) Equation

Let us consider the following coupled (2+1) dimensional BKP equation [29-32] as

$$\left.\begin{aligned} u_y &= w_x \\ v_x &= w_y \\ w_t &= w_{xxx} + w_{yyy} + 6(wu)_x + 6(wv)_y \end{aligned}\right\} \tag{18}$$

Here $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ presents the evolutions the surface in any natural varied instances. The coupled equation (18) is also appeared as a model equation for describing various types of physical phenomena in mathematical physics. Many research scholar in mention in refs. [29-31] have been derived the (2+1) dimensional BKP equation. From the articles in mention ref. [32, 36], we have also observed that the reduce ODE (4.7) is incorrect (sec more details in Appendix 1 and 2).

To get the new valuable explicit form solutions to the equations (18), we introduce the compound variables *x, y,* and *t* to a single variable as

$$u(x,y,t) = u(\xi),\ v(x,y,t) = v(\xi),\quad w(x,y,t) = w(\xi),\quad \xi = x + y - V\,t\,, \tag{19}$$

where V is an arbitrary constants. By substituting eq. (19) into eq. (18), we obtain the following coupled nonlinear ordinary differential equations:

$$\left.\begin{aligned} u' - w' &= 0 \\ v' - w' = 0 &= 0 \\ 2w''' + Vw' + 6(uw)' + 6(vw)' &= 0 \end{aligned}\right\}. \tag{20}$$

Integrating the ODEs (20) and neglecting the constants of integration for convenience, we obtain

$$\left.\begin{aligned} u - w &= 0 \\ v - w = 0 &= 0 \\ 2w'' + Vw + 6uw + 6vw &= 0 \end{aligned}\right\}. \tag{21}$$

Therefore, the solutions of the ODEs (21) can be

represented according to the advance $\exp(-\varphi(\xi))$-expansion method as

$$\left.\begin{aligned} u(\xi) &= a_0 + a_1 e^{-\varphi(\xi)} + a_2 \left(e^{-\varphi(\xi)}\right)^2, \ a_2 \neq 0 \\ v(\xi) &= b_0 + b_1 e^{-\varphi(\xi)} + b_2 \left(e^{-\varphi(\xi)}\right)^2, \ b_2 \neq 0 \\ v(\xi) &= c_0 + c_1 e^{-\varphi(\xi)} + c_2 \left(e^{-\varphi(\xi)}\right)^2, \ c_2 \neq 0 \end{aligned}\right\} \quad (22)$$

where a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1 and c_2 are unknowns constants to be determined later. By substituting eq. (22) in the eq. (21) and collecting all terms with same power of the coefficient of $e^{-\varphi(\xi)}$ together, we obtain a system of algebraic equations. The system of algebraic equations is overlooked for convenience. Solving the resulting algebraic equations, we obtain the following set of solutions:

Set 1:

$$\left\{V = -8\lambda\mu,\ a_0 = b_0 = c_0 = -\frac{1}{3}\lambda\mu, a_1 = b_1 = c_1 = 0,\ a_2 = b_2 = c_2 = -\mu^2\right\} \quad (23)$$

Set 2:

$$\left\{V = 8\lambda\mu,\ a_0 = b_0 = c_0 = -\lambda\mu, a_1 = b_1 = c_1 = 0,\ a_2 = b_2 = c_2 = -\mu^2\right\} \quad (24)$$

According to Set 1 and Set 2 and the general solutions of (4), the traveling wave solutions of BKP equation (18) are obtained in the following form:

$$u_1(x,y,t) = v_1(x,y,t) = w_1(x,y,t) = -\frac{1}{3}\lambda\mu - \lambda\mu \tan^2\left(\sqrt{\lambda\mu}\ (x + y + 8\lambda\mu t + \xi_0)\right),\ \lambda\mu > 0, \quad (25)$$

$$u_2(x,y,t) = v_2(x,y,t) = w_2(x,y,t) = -\frac{1}{3}\lambda\mu - \lambda\mu \cot^2\left(\sqrt{\lambda\mu}\ (x + y + 8\lambda\mu t + \xi_0)\right),\ \lambda\mu > 0, \quad (26)$$

$$u_3(x,y,t) = v_3(x,y,t) = w_3(x,y,t) = -\frac{1}{3}\lambda\mu + \lambda\mu \tanh^2\left(\sqrt{-\lambda\mu}\ (x + y + 8\lambda\mu t + \xi_0)\right),\ \lambda\mu < 0, \quad (27)$$

$$u_4(x,y,t) = v_4(x,y,t) = w_4(x,y,t) = -\frac{1}{3}\lambda\mu + \lambda\mu \coth^2\left(\sqrt{-\lambda\mu}\,(x + y + 8\lambda\mu t + \xi_0)\right),\ \lambda\mu < 0, \quad (28)$$

$$u_5(x,y,t) = v_5(x,y,t) = w_5(x,y,t) = -\lambda\mu \sec^2\left(\sqrt{\lambda\mu}\,(x + y - 8\lambda\mu t + \xi_0)\right),\ \lambda\mu > 0, \quad (29)$$

$$u_6(x,y,t) = v_6(x,y,t) = w_6(x,y,t) = -\lambda\mu \cos ec^2\left(\sqrt{\lambda\mu}\,(x + y - 8\lambda\mu t + \xi_0)\right),\ \lambda\mu > 0, \quad (30)$$

$$u_7(x,y,t) = v_7(x,y,t) = w_7(x,y,t) = -\lambda\mu \sec h^2\left(\sqrt{-\lambda\mu}\,(x + y - 8\lambda\mu t + \xi_0)\right),\ \lambda\mu < 0, \quad (31)$$

$$u_8(x,y,t) = v_8(x,y,t) = w_8(x,y,t) = -\lambda\mu - \lambda\mu \coth^2\left(\sqrt{-\lambda\mu}\ (x + y - 8\lambda\mu t + \xi_0)\right),\ \lambda\mu < 0, \quad (32)$$

From the solutions (25) to (32), we observe that the proposed advance $\exp(-\Phi(\xi))$-expansion scheme according to subsidiary equation (6) gives more valuable explicit form solutions to the BKP equation. These solutions would be more useful to describe physical mechanism of natural phenomena in applied sciences or any varied natural instances, where the standard BKP equation is applicable. The obtaining solutions are also gives various types of solitary and periodic wave solutions according to the variation of the additional free parameters. Some of the important solitary and periodic wave solutions are described and presented in graphically. Solutions $u_1(x,y,t)$, $v_1(x,y,t)$, $w_1(x,y,t)$, $u_2(x,y,t)$, $v_2(x,y,t)$ and $w_2(x,y,t)$ are presented the exact periodic wave solutions according to the fixed values $\lambda = 0.5$, $\mu = 1$, $\xi_0 = 0.5$, and $y = 0$. The shape of the periodic solution is shown in Figure 4. Solutions $u_3(x,y,t)$, $v_3(x,y,t)$, $w_3(x,y,t)$ and $u_4(x,y,t)$, $v_4(x,y,t)$, $w_4(x,y,t)$ are represented the solitary wave solutions of peakon type and cupson type corresponding to the values $\lambda = -0.25$, $\mu = 0.5$, $\xi_0 = 0.5$ and $y = 0$ respectively. The solitary wave solutions of peakon type and soliton type are shown in Figure 5 (a) and 5 (b) respectively. Solutions $u_5(x,y,t)$, $v_5(x,y,t)$, $w_5(x,y,t)$ and $u_6(x,y,t)$, $v_6(x,y,t)$, $w_6(x,y,t)$ are presented the solitary wave solutions of compacton type and singular soliton type corresponding to the values $\lambda = 0.25$, $\mu = 0.5$,

$\xi_0 = 0$ and $y = 0$ respectively. Compacton is a new category of solitons with compact spatial hold up such that each compacton is a soliton restricted to a finite core. It has remarkable soliton property that after colliding with other compactons, they come back with the same coherent shape. The solitary wave solutions of compacton type and singular soliton type are shown in Figure 6 (a) and 6 (b) respectively. Solutions $u_7(x,y,t)$, $v_7(x,y,t)$, $w_7(x,y,t)$ and $u_8(x,y,t)$, $v_8(x,y,t)$, $w_8(x,y,t)$ are presented the solitary wave solutions of non-topological bell type and singular soliton type corresponding to the values $\lambda = -0.25$, $\mu = 0.5$, $\xi_0 = 0$ and $y = 0$ respectively. The shape of the non-topological bell type solitary wave solution obtain from the solution $u_7(x,y,t)$ is shown in Figure 7. Other figures are overlooked for convenience.

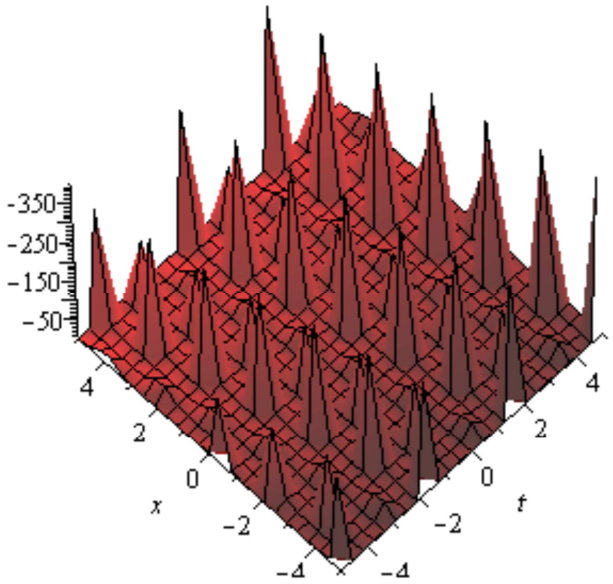
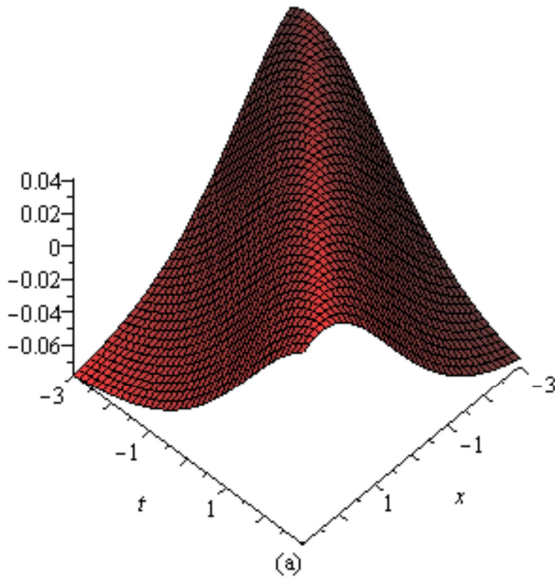


Figure 4. *Shape of exact peeriodic wave solutions of the solution* $u_1(x,y,t)$ *with* $y = 0$ *and* $-5 \le x,t \le 5$.



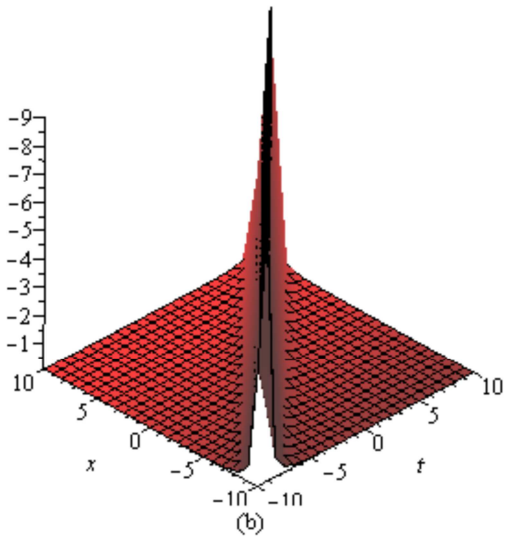
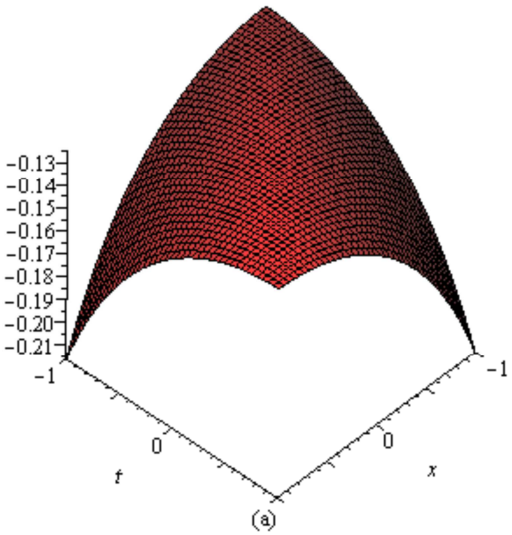


Figure 5. *Shape of exact solitary wave solutions of (a) peakon nature for* $u_3(x,y,t)$ *with* $-3 \le x,t \le 3$ *and (b) cupson nature for* $u_4(x,y,t)$ *with* $-10 \le x,t \le 10$.



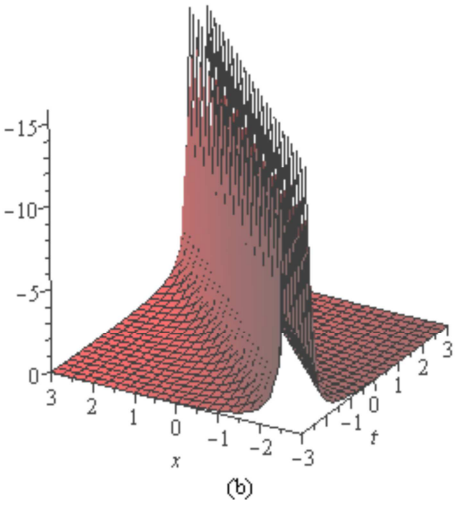


Figure 6. *Shape of exact solitary wave solutions of (a) compacton nature for* $u_5(x,y,t)$ *with* $-1 \le x,t \le 1$ *and (b) soliton nature for* $u_6(x,y,t)$ *with* $-3 \le x,t \le 3$.

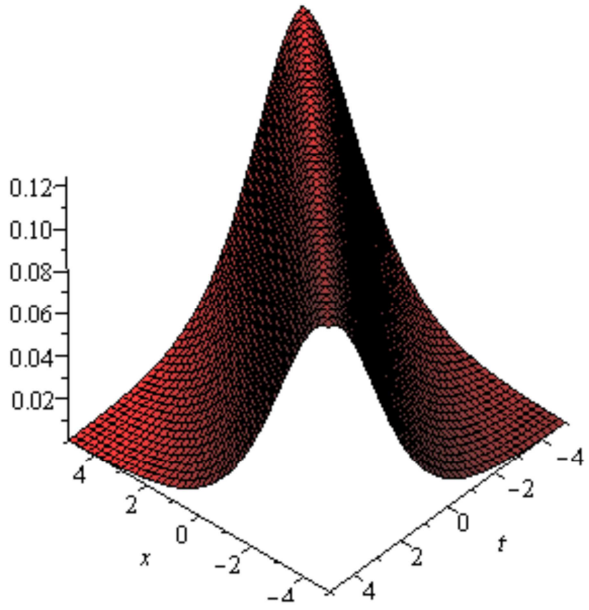


Figure 7. *Shape of exact non-topological bell nature solitary wave solution of* $u_7(x,y,t)$ *with* $-10 \le x,t \le 10$.

4. Advantages and Comparison

From the references [21-24], we observe that the exp (-Φ(ξ))-expansion method does not give rich solutions to comprehend inner picture to the NLEEs. Therefore, in order to get more traveling wave solutions and to understand the inner structure apparently of the nonlinear physical phenomena, in this article we choose the ODE (4) as auxiliary equation and their solutions have been used. The main advantage of the introduced method is that it offers more general valuable and huge amount of exact traveling wave solutions with some free parameters. The exact solutions have its extensive importance to interpret the inner structures of the natural phenomena in mathematical physics, chemistry and biology. The explicit solutions represented various types of solitary wave solutions according to the variation of the physical parameters. In this article, various types of solitary and periodic wave solutions are presented graphically in Fig. 1 to 7. This method not only re-derives all known solutions in a systemic way but also obtains some entirely new and more explicit solutions to the nonlinear NLEEs. The algebraic manipulation of this method with the help of Maple is much easier than the other existing method. Sometimes the method gives solutions in disguised versions of known solutions that may be found by other methods.

Many scholar applied different methods for finding the traveling wave solutions to the coupled classical Boussineq equation and the BKP equation. If we put $\lambda = 1$, then the solutions $u_3(x,t)$ and $v_3(x,t)$ of the classical Boussineq equation are equal to the solution $u(x,t) = \left\{ a_0 + \frac{2}{\sqrt{3}} \sigma \sqrt{-\mu} \tanh\left[\sqrt{-\mu} \left(x - a_0 t + \xi_0 \right) \right] \right\}$ and $v(x,t) = \left\{ -\frac{2\mu}{3} \sec h^2 \left[\sqrt{-\mu} \left(x - a_0 t + \xi_0 \right) \right] - 1 \right\}$ with $\sigma = 1$ which are found in the article [28] and solutions $u_4(x,t)$ and $v_4(x,t)$ of the classical Boussineq equation are also equal to the solution $u(x,t) = \left\{ a_0 + \frac{2}{\sqrt{3}} \sigma \sqrt{-\mu} \coth\left[\sqrt{-\mu} \left(x - a_0 t + \xi_0 \right) \right] \right\}$ and $v(x,t) = \left\{ \frac{2\mu}{3} \csc h^2 \left[\sqrt{-\mu} \left(x - a_0 t + \xi_0 \right) \right] - 1 \right\}$ with $\sigma = 1$ which are also found in article [28]. The solutions $u_8(\xi) = v_8(\xi) = w_8(\xi)$ which are found in the article [32] are equivalent to our obtain solutions $u_7(x,y,t) = v_7(x,y,t) = w_7(x,y,t)$. It is worth mentioning that the (G'/G)-expansion method is special case of the extended G'/G-expansion method. N. A. Kudryashov [34] have been investigated that the (G'/G)-expansion method is equivalent to the well known tanh-method. So, comparison our obtain solution with the (G'/G)-expansion method and the extended (G'/G)-expansion method is sufficient. Beside this, we achieved our solutions via the advance exp(-Φ(ξ))-expansion method with the auxiliary ODE $\Phi'(\xi) + \lambda \exp(\Phi(\xi)) + \mu \exp(-\Phi(\xi)) = 0$ while the (G'/G)-expansion method and the extended (G'/G)-expansion method performed with others. It noteworthy to point out that some of our solutions are coincided with already published results, if the parameters taken particular values which authenticate our solutions. Therefore, it can be decided that the proposed method is powerful mathematical tool for solving nonlinear evolutions equations and all kinds of NLEEs can be solved through this method.

5. Conclusions

The advance exp(-Φ(ξ)) -expansion method has been successfully implemented to construct new generalized traveling wave solutions to the classical Boussineq equation and the (2+1) dimensional BKP equation. The obtained solutions in this article are expressed by the hyperbolic functions, trigonometric functions and rational functions. We have noted that the advance exp(-Φ(ξ)) -expansion method changes the given difficult problems into simple problems which can be solved easily. We hope this method can be more effectively used to many others NLEEs arising in mathematical physics and engineering. The graphical representations explicitly reveal the high applicability and competence of the proposed algorithm.

Appendix

Appendix 1

Zheng [32] have converted the (2+1) dimensional Boussinesq and Kadomtsev-Petviashili equation to ODEs as follows:

Now, we consider the (2+1) dimensional Boussinesq and Kadomtsev-Petviashili equation:

$$u_y = w_x \tag{33}$$

$$v_x = w_y \tag{34}$$

$$q_t = q_{xxx} + q_{yyy} + 6(qu)_x + 6(qv)_y \tag{35}$$

In order to obtain the traveling wave solutions of (33), (34) and (35), we suppose that

$$u(x,y,t) = u(\xi),\ v(x,y,t) = v(\xi),\quad w(x,y,t) = w(\xi),\quad \xi = x + y - c\ t\ , \tag{36}$$

a, d, c are constants that to be determined later.

By using the wave variable (36), Eqs. (33)-(35) can be converted into ODEs into eq. (18), we obtain the following coupled nonlinear ordinary differential equations:

$$u' - q' = 0 \tag{37}$$

$$v' - q' = 0 \tag{38}$$

$$2q''' - cq' - 6uq' - 6qu' - 6vq' - 6qv' = 0 \tag{39}$$

The converted ODE (39) may be incorrect.

Appendix 2

From the article [36], we have also observed that the converted ODEs

$$\left.\begin{aligned} &U - W = 0 \\ &V - W = 0 = 0 \\ &2W_{\xi\xi} - cW - 6UW - 6VW = 0 \end{aligned}\right\} \tag{40}$$

according to wave variable transformations $U(x,y,t) = U(\xi),\quad V(x,y,t) = V(\xi),\quad W(x,y,t) = W(\xi)$ where, $\xi = x + y - c\ t$ may be incorrect.

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