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# Hamiltonicity of Mycielski Graphs

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**Abstract:** Fisher, McKenna, and Boyer showed that if a graph $G$ is hamiltonian, then its Mycielski graph $\mu(G)$ is hamitonian. In this note, it was shown that for a bipartite graph $G$, if its mycielski graph $\mu(G)$ is hamiltonian, then $G$ has a Hamilton path.

**Keywords:** Bipartite Graphs, Mycielski Graph, Hamilton Cycle, Walk

## 1. Introduction

All graphs considered in this paper are finite and simple. For notation and terminology not defined here, we refer to West [16]. Mycielski [13] found a kind of construction to create triangle-free graphs with large chromatic numbers. For a graph $G$ on vertices $V=\{v_1,v_2,\cdots,v_n\}$, the Mycielski graph $\mu(G)$ of $G$ is defined as

$$V(\mu(G))=X\cup Y\cup\{z\}=\{x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n,z\}$$

$z$ is adjacent to every $y_i$, and

if $v_iv_j\in E(G)$, then $x_ix_j, y_iy_j\in E(\mu(G))$.

For example $\mu(K_2)=C_5$, an $\mu^2(K_2)$ is known as the Grötsch graph, see Figure 1. Mycielski showed that $\mu^k(K_2)$ is triangle-free and has chromatic number $k+2$. In general, for a graph $G$ (not necessarily a triangle-free graph), $\chi(\mu(G))=\chi(G)+1$, where $\chi(G)$ denotes the chromatic number of $G$.

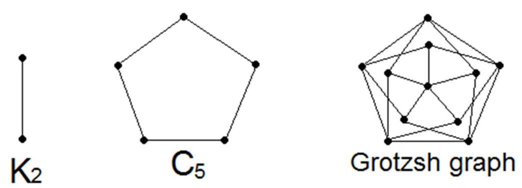


***Figure 1.*** *$\mu(K_2)=C_5$ and Grötsch graph.*

Recently, a number of papers are devoted to the various parameters of Mycielski graphs, such as the fractional chromatic number [9], circular chromatic number [2, 5, 8, 10, 12], connectivity [1, 6]. Total chromatic number [3], hub number [11], covering number [14], total weight choosability [15], Hamilton-connectivity [7].

A spanning cycle (resp. path) of a graph is called Hamilton cycle (resp. Hamiton path). A graph is said to be *hamiltonian* if it has a Hamilton cycle. In general, it is NP-hard to decide whether a graph has a Hamilton cycle or not. Our main objective is to search best possible condition for a graph $G$ whose Mycielski graph is hamiltonian. Fisher, McKenna, and Boyer [4] obtained the following results.

Theorem 1.1. ([4]) If $G$ is hamiltonian, then $\mu(G)$ is hamiltonian.

Theorem 1.2. ([4]) If $G$ is not connected, then $\mu(G)$ is not hamiltonian.

Theorem 1.3. ([4]) If $G$ has at least two pendent vertices, then $\mu(G)$ is not hamiltonian.

One might suspect that the converse of Theorem 1.1 is true.

Conjecture 1.4. For a graph $G$, if $\mu(G)$ is hamiltonian, then $G$ has a Hamilton cycle.

But, the conjecture is not true, see the graph $\Theta_{4,4,4}$ in Figure 2. Clearly, $\mu(\Theta_{4,4,4})$ is not hamiltonian by Lemma 2.1. However, a Hamilton cycle of $\mu(\Theta_{4,4,4})$ is given.

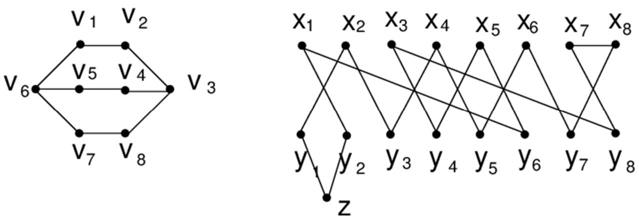


***Figure 2.*** *$\Theta_{4,4,4}$ and a Hamilton cycle of $\mu(\Theta_{4,4,4})$.*

In this note, we show the converse of Theorem 1.1 is almost true for bipartite graphs. Precisely, we show that for a biparite graph $G$, if the mycielski graph $\mu(G)$ of a graph $G$ is hamiltonian, then $G$ has a Hamilton path.

## 2. Main Theorem

For a graph $G$, $c(G)$ denotes the number of components of $G$. The following is the well-known 1-tough condition for the hamiltonicity of graphs.

Lemma 2.1. If $G$ is hamiltonian, then for any nonempty subset $S \subseteq V(G)$, $c(G-S) \leq |S|$.

A bipartite graph $G = G[B,W]$ is said to be *balanced* if $|B| = |W|$.

Lemma 2.2. Let $G$ be a bipartite graph. If $\mu(G)$ is hamiltonian, then $G$ is balanced.

*proof.* Let $(B,W)$ be the bipartition of $G$, and let $B', W'$ be the corresponding vertices in $\mu(G)$, and $z$ be the special vertex. By contradiction, suppose that $G$ is not balanced, and that $|B| > |W|$. Take $S = \{z\} \cup W \cup W'$. Then $c(\mu(G) - S) = |B \cup B'| = 2|B| > 2|W| + 1$, which contradicts the fact of Lemma 2.1. This shows that $|B| = |W|$.

Lemma 2.3. For a graph $G$, $\mu(G)$ is hamiltonian if and only if $G$ has a walk $P$ with two distinct end vertices $v_1$ and $v_2$, say, with the following additional properties:

(1) every vertex of $G$ appear precisely twice on $P$, and

(2) there exists an edge e of $P$, such that $P-e$ is divided into two walks $P_1$ and $P_2$, and if for every vertex $v \in V(G)$, if $v$ appear twice on some $P_i$, then the length of $P[v,v]$ is odd; and otherwise, the length of $P[v,v]$ is even.

*Proof.* First we show the necessity. Let $C$ be a Hamilton cycle of $\mu(G)$. Then $C-z$ is a path of $\mu(G)$ joining two vertices $y_1$, $y_2$, say, in $Y$. Recall that $y_1$ and $y_2$ correspond to $v_1$ and $v_2$ in $G$, respectively.

Walking along on the path $C-z$ from $y_1$ to $y_2$, one obtain a walk $P$ in $G$, with the property that *(1)* every vertex of $G$ appear twice on $P$. Let $e$ be the unique edge of $P$, which joins two vertices in $X$, and let $P_1$ and $P_2$ be the two walks results from $P$ deleting $e$. Let $v$ be a vertex of $G$, and assume that $v$ corresponds to a vertex $x$ of $X$ and a vertex $y$ of $Y$. Then the walk $P[v,v]$ corresponds to the path of $(C-z)[x,y]$, and thus they have the same length. By the definition of Mycielski graph, $Y$ is an independent set of $\mu(G)$, the length of $(C-z)[x,y]$ is odd if and only if $v$ appear twice either on $P_1$ or $P_2$. This proves *(2)*.

If there is a walk in $G$ with the property in the assumption of the lemma, one could find a Hamilton path in $\mu(G)-z$ with two end vertices in $Y$ by going along on the walk, and thereby connecting them to $z$, we are able to obtain a Hamilton cycle of $\mu(G)$.

Now we are ready to present our main theorem.

Theorem 2.4. *Let $G$ be a bipartite graph. If $\mu(G)$ is hamiltonian, then $G$ has a Hamilton path.*

*Proof.* Let $C$ be a Hamilton cycle of $\mu(G)$. Then, as in proof of Lemma 2.3, one could get a walk $P$ with the properties described in the assumption in Lemma 2.3. Let $e, P_1, P_2$ be those, as given in Lemma 2.3. We claim that both $P_1$ and $P_2$ are Hamilton paths of $G$. Note that there is no vertex of $G$ appear on $P_1$ or $P_2$ twice. Since, otherwise, by Lemma 2.3, the length of $P_i[v,v]$ is odd if there is such a vertex $v$ in $G$. It implies that $G$ has an odd cycle, which contradicts our assumption that $G$ is bipartite. Since every vertex of $G$ appear on $P$ twice, and by our claim that every vertex of $G$ appear on $P_i$ at most once, we conclude that every vertex of $G$ appear on $P_i$ precisely once, and thus both $P_1$ and $P_2$ are Hamilton paths of $G$.

## 3. Conclusion

In this short note, it was shown for a biparite graph $G$, if the mycielski graph $\mu(G)$ of a graph $G$ is hamiltonian, then $G$ has a Hamilton path. We conclude with posing the following two conjectures.

Conjecture 3.1. *For a graph $G$, if $\mu(G)$ is hamiltonian, then $G$ has a Hamilton path.*

*For a positive integer $k$, let us define $\mu^k(G) = \mu(\mu^{k-1}(G))$, where $\mu^1(G) = \mu(G)$.*

Conjecture 3.2. *For a graph $G$ without isolated vertices, there exists a natural number $k_0$, such that $\mu^k(G)$ is hamiltonian for all $k \geq k_0$.*

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