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Application of the Differential Transform Method for the Nonlinear Differential Equations

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Abstract: This paper aims to find analytical solutions of some analytical solutions of some non-linear differential equations using a new integral transform "Aboodh transform" with the differential transform method. The nonlinear terms can be easily handled by the use of differential transform method. This method is more efficient and easy to handle such differential equations in comparison to other methods. The results reveal that this method is very efficient, simple and can be applied to other nonlinear problems

Keywords: Aboodh Transform, Differential Transform Method, Nonlinear Differential Equations

1. Introduction

Many physical problems can be described by mathematical models that involve ordinary or partial differential equations. A mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximation solution helps us to understand the means of these mathematical models. Several numerical methods were developed for solving ordinary or partial differential equations. In recent years, some researchers used many powerful methods for obtaining exact solutions of nonlinear partial differential equations, such as homotopy perturbation method [1-2], modified variational iteration method [3], non-perturbative methods [4], Adomian Decomposition Method (ADM) [5-6], and reduced differential transform method [7]. the differential transform method has been developed for solving the differential and integral equations. For example in [8] this method is used for solving a system of differential equations and in [9] for differential-algebraic equations. In [10-13] this method is applied to partial differential equations and in [14-16] to one- dimensional Volterra integral and integro-differential equations.

New integral transform “Aboodh transform” [17-23] is particularly useful for finding solutions for these problems. Aboodh transform is a useful technique for solving linear Differential equations but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. This paper is using differential transforms method to decompose the nonlinear term, so that the solution can be obtained by iteration procedure. This means that we can use both Aboodh transform and differential transform methods to solve many nonlinear problems.

2. Aboodh Transform

Definition:

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A, defined by:

$$A = \{f(t): \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\} \qquad (1)$$

For a given function in the set M must be finite number, k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}\,dt, t \geq 0, \mathrm{k}_1 \leq v \leq \mathrm{k}_2 \quad (2)$$

Theorem (1)

Let $K(v)$ be Aboodh transform of $f(t), A[f(t)] = K(v)$ then:

$$\text{(i) } A[f'(t)] = vK(v) - \frac{f(0)}{v},$$

$$\text{(ii) } A[f''(t)] = v^2K(v) - \frac{f'(0)}{v} - f(0)$$

$$\text{(iii) } A\left[f^{(n)}(t)\right] = v^nK(v) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}}. \quad (3)$$

Proof

By the definition we have:

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}\,dt,$$

Integrating by parts, we get

$$A\left[\frac{\partial f}{\partial t}(x,t)\right] = \int_0^\infty \frac{1}{v}\frac{\partial f}{\partial t}e^{-vt}\,dt = \lim_{p\to\infty}\int_0^p \frac{1}{v}\frac{\partial f}{\partial t}e^{-vt}\,dt$$

$$= \lim_{p\to\infty}\left\{\left[\frac{1}{v}e^{-vt}\right]_0^p + \int_0^p e^{-vt}f(x,t)\,dt\right\}$$

$$= vK(x,v) - \frac{f(x,0)}{v}$$

3. Differential Transform

Differential transform of the function $Y(x)$ for the k-derivative is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \quad (4)$$

Where $Y(x)$ is original function and $y(k)$ is the transformed function.

And the inverse differential transform of $Y(k)$ is defined as:

$$Y(x) = \sum_{k=0}^{\infty} y(k)x^k$$

The main theorems of the one – dimensional differential transform are.

Theorem (2): If $w(k) = y(x) \pm z(x)$, then $W(k) = Y(k) \pm Z(k)$

Theorem (3): If $w(x) = cy(x)$, Then $W(k) = cY(k)$

Theorem (4): If $w(x) = \frac{d\,y(x)}{dx}$ then $W(k) = (k+1)y(k+1)$

Theorem (5): If $w(x) = \frac{d^n y(x)}{dx^n}$ = then $W(k) = \frac{(k+n)!}{k!}y(k+n)$

Theorem (6): If $w(x) = y(x)z(x)$, then $W(k) = \sum_{r=0}^{k} y(r)Z(k-r)$

Theorem (7): If $w(x) = x^n$ then $W(k) = \delta(k-n) = \begin{cases} 1, k = n \\ 0, k \neq n \end{cases}$

Note that *c* is a constant and *n* is a nonnegative integer.

4. Analysis of Differential Transform

In this section, we will introduce a reliable and efficient algorithm to calculate the differential transform of nonlinear functions.

I / Exponential nonlinearity: $f(y) = e^{ay}$

From the definition of transform

$$F(0) = \left[e^{ay(x)}\right]_{x=0} = e^{ay(0)} = e^{aY(0)} \quad (5)$$

Taking a differential of $f(y) = e^{ay}$ with respect to *x*, we get:

$$\frac{df(y)}{dx} = ae^{ay}\frac{dy(x)}{dx} = af(y)\frac{dy(x)}{dx} \quad (6)$$

Application of the differential transform to Eq (6) gives:

$$(k+1)y(k+1) = a\sum_{m=0}^{k}(m+1)Y(m+1)F(k-m) \quad (7)$$

Replacing *k* +1 by *k* gives

$$F(k) = a\sum_{m=0}^{k-1}\left(\frac{m+1}{k}\right)Y(m+1)F(k-1-m) \quad k \geq 1 \quad (8)$$

Then from Eqs (5) and (8), we obtain the recursive relation

$$F(k) = \begin{cases} e^{aY(0)}, k = 0 \\ a\sum_{m=0}^{k-1}\left(\frac{m+1}{k}\right)Y(m+1)F(k-1-m), k \geq 1 \end{cases} \quad (9)$$

II / Logarithmic nonlinearity: $f(y) = ln(a+by)$, $a + by > 0$.

Differentiating $f(y) = ln(a+by)$, with respect to *x*, we get:

$$\frac{df(y(x))}{dx} = \frac{b}{a+by}\frac{dy(x)}{dx} \text{ , or } a\frac{df(y)}{dx} = b\left[\frac{dy(x)}{dx} - y\frac{df(y)}{dx}\right] \quad (10)$$

By the definition of transform:

$$F(0) = [ln(a+by(x))]_{x=0} = ln(a+by(0)) = ln(a+bY(0)) \quad (11)$$

Take the differential transform of Eq.(10) to get:

$$aF(k+1) = b\left[Y(k+1) - \sum_{m=0}^{k}\frac{m+1}{k+1}F(m+1)Y(k-m)\right] \quad (12)$$

Replacing *k* +1 by *k* yields:

$$aF(k) = b\left[Y(k) - \sum_{m=0}^{k-1}\frac{m+1}{k}F(m+1)Y(k-1-m)\right], \ k \geq 1 \quad (13)$$

Put *k* =1 into Eq.(13) to get:

$$F(1) = \frac{b}{a+bY(0)}Y(1) \quad (14)$$

For $k \geq 2$, Eq. (13) can be rewritten as

$$F(k) = \frac{b}{a+bY(0)}\left[Y(k) - \sum_{m=0}^{k-2}\frac{m+1}{k}F(m+1)Y(k-1-m)\right] \quad (15)$$

Thus the recursive relation is:

$$F(k)=\begin{cases} ln(a+by(0)), k=0 \\ \frac{b}{a+bY(0)}Y(1), k=1 \\ \frac{b}{a+bY(0)}\left[Y(k)-\sum_{m=0}^{k-2}\frac{m+1}{k}F(m+1)Y(k-1-m)\right], k\geq 2 \end{cases}$$

5. Application

In this section we solve some nonlinear differential equation by combine Aboodh transform and differential transform method

Example (1)

Consider the simple nonlinear first order differential equation.

$$y'=y^2 \quad y(0)=1 \tag{16}$$

First applying Aboodh transform on both sides to find:

$$vk(v)-\frac{f(0)}{v}=T(y^2)$$

$$k(v)=\frac{1}{v^2}+\frac{1}{v}T(y^2) \tag{17}$$

$k(v)$ is the Aboodh transform of $y(t)$,

The standard Aboodh transformation method defines the solution $y(t)$ by the series.

$$y=\sum_{n=0}^{\infty}y(n) \tag{18}$$

Operating with Aboodh inverse on both sides of Eq (17) gives:

$$y(t)=1+A^{-1}\left[\frac{1}{v}A(y^2)\right] \tag{19}$$

Substituting Eq (18) into Eq (19) we find:

$$y(n+1)=A^{-1}\left[\frac{1}{v}A(A_n)\right] \qquad n\geq 0 \tag{20}$$

Where $y(0)=1$, $A_n=\sum_{m=0}^{k}Y(r)F(n-r)$, and $A_0=1$

For n = 0, we have:

$$y(1)=A^{-1}\left[\frac{1}{v}A(A_0)\right]=A^{-1}\left[\frac{1}{v}A(A_0)\right]=t$$

For n =1, we have:

$$A_0=2t \text{ and } y(2)=A^{-1}\left[\frac{1}{v}T(A_1)\right]=A^{-1}\left[\frac{1}{v}A(2t)\right]=t^2$$

For n = 2, we haves:

$$A_0=3t^2 \text{ and } y(3)=A^{-1}\left[\frac{1}{v}A(A_2)\right]=A^{-1}\left[\frac{1}{v}A(3t^2)\right]=t^3$$

The solution in *a* series form is given by.

$$y(t)=y(0)+y(1)+\ y(2)+y(3)+\cdots.$$

$$y(t)=1+t+\ t^2+t^3+\cdots.=\frac{1}{1-t}$$

Example (2)

We consider the following nonlinear differential equation.

$$\frac{dy}{dt}=y-y^2 \quad y(0)=2 \tag{21}$$

In a similar way we have:

$$vk(v)-\frac{f(0)}{v}=A(y-y^2)$$

$$k(v)=\frac{2}{v^2}+\frac{1}{v}A(y-y^2) \tag{22}$$

The inverse of Aboodh transform implies that:

$$y(t)=2+A^{-1}\left[\frac{1}{v}A(y-y^2)\right] \tag{23}$$

The recursive relation is given by:

$$y(n+1)=A^{-1}\left[\frac{1}{v}A(y(n)-A_n)\right] \qquad n\geq 0 \tag{24}$$

Where $y(0)=2$, and $A_n=\sum_{m=0}^{n}Y(r)y(n-r)$

The first few components of A_n are

$$A_0=y^2(0), A_1=2y\ (0)y(1)\ , A_2=2y\ (0)y(2)+y^2(1)\ ,$$

$$A_3=2y\ (0)y(3)+2y\ (1)y(2)\ ,......$$

From the recursive relation we have:

$$y(0)=2, A_0=4$$

$$y(1)=A^{-1}\left[\frac{1}{v}A(y(0)-A_0)\right]=A^{-1}\left[\frac{1}{v}A(-2)\right]=-2t$$

$$y(2)=A^{-1}\left[\frac{1}{v}A(y(1)-A_1)\right]=A^{-1}\left[\frac{1}{v}A(6t)\right]=3t^2$$

$$y(3)=A^{-1}\left[\frac{1}{v}A(y(2)-A_2)\right]=A^{-1}\left[\frac{1}{v}A(-13t^2)\right]=-\frac{13}{3}t^3$$

Then we have the following approximate solution to the initial problem.

$$y(t)=y(0)+y(1)+\ y(2)+y(3)+\cdots.$$

$$y(t)=2-2t+3\,t^2-\frac{13}{3}t^3+\frac{25}{4}t^4+\cdots.=\frac{2}{2-e^{-t}}$$

Example (3)

Consider the nonlinear initial – value Problem

$$y''(x)=2y+4yln\ y\ y>0 \quad y(0)=1, y'(0)=0 \tag{25}$$

Applying Aboodh transform to Eq (25) and using the initial conditions, we obtain.

$$v^2k(v)-\frac{f'(0)}{v}-f(0)=A(2y+4yln\ y)$$

$$k(v) = \frac{1}{v^2} + \frac{1}{v^2} A(2y + 4yln\, y) \quad (26)$$

Take the inverse of Eq (26) to find:

$$y(t) = 1 + A^{-1}\left[\frac{1}{v^2} A(2y + 4yln\, y)\right] \quad (27)$$

The recursive relation is given by:

$$y(n+1) = A^{-1}\left[\frac{1}{v^2} A(2y + 4yln\, y)\right] \quad (28)$$

Where $A_n = \sum_{m=0}^{n} Y(m)F(n-m)$ and $y(0) = 1$ (29)
And

$$F(n) = \begin{cases} ln(y(0)), n = 0 \\ \frac{y(1)}{y(0)}, n = 1 \\ \frac{y(n)}{y(0)} + \sum_{m=0}^{n-2} \frac{m+1}{n\, y(0)} F(m+1) y(n-1-m), n \geq 2 \end{cases} \quad (30)$$

Then we have:

$F(0) = 0, A_0 = 0$, and $y(1) = A^{-1}\left[\frac{1}{v^2} A(2)\right] = A^{-1}\left[\frac{2}{v^4}\right] = x^2$

$F(1) = x^2, A_1 = x^2$, and $y(2) = A^{-1}\left[\frac{1}{v^2} A(6x^2)\right] = A^{-1}\left[\frac{12}{v^6}\right] = \frac{x^4}{2}$

$F(2) = 0, A_2 = x^4$, and $y(3) = A^{-1}\left[\frac{1}{v^2} A(5x^4)\right] = A^{-1}\left[\frac{120}{v^8}\right] = \frac{x^6}{6}$

Then the exact solution is:

$$y(x) = y(0) + y(1) + y(2) + y(3) + \cdots.$$

$$y(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = e^{x^2}$$

Example (4)
Consider the initial –value problem of Bratu-type.

$$y''(x) - 2e^y = 0 \;\; 1 < x < 0 \;\; y(0) = y'(0) = 0, \quad (31)$$

Take Aboodh transform of this equation and use the initial condition to obtain:

$$v^2 k(v) - \frac{f'(0)}{v} - f(0) = A(2e^y)$$

$$k(v) = \frac{1}{v^2} A(2e^y) \quad (32)$$

Take the inverse to obtain:

$$y(t) = A^{-1}\left[\frac{1}{v^2} A(2e^y)\right]$$

Then the recursive relation is given by:

$$y(n+1) = A^{-1}\left[\frac{1}{v^2} A(2F(n))\right] y(0) = 0 \quad (33)$$

Where $y(0) = 0$ and

$$F(n) = \begin{cases} y(0), n = 0 \\ \sum_{m=0}^{n-1} \frac{m+1}{n} Y(m+1) F(n-m-1), n \geq 1 \end{cases} \quad (34)$$

Then from Eqs (33) and (34) we have

$F(0) = 1$ and $y(1) = A^{-1}\left[\frac{1}{v^2} A(2)\right] = A^{-1}\left[\frac{2}{v^4}\right] = x^2$

$F(1) = x^2$, $A_1 = x^2$, and $y(2) = A^{-1}\left[\frac{1}{v^2} A(2x^2)\right] = A^{-1}\left[\frac{4}{v^6}\right] = \frac{x^4}{6}$

$F(2) = \frac{2}{3}x^4$, $A_2 = x^4$, and $y(3) = A^{-1}\left[\frac{1}{v^2} A(\frac{4}{3}x^4)\right] = A^{-1}\left[\frac{32}{v^8}\right] = \frac{2}{45}x^6$

Then the series solution is

$$y(x) = y(0) + y(1) + y(2) + y(3) + \cdots.$$

$$y(x) = 1 + x^2 + \frac{x^4}{6} + \frac{2}{45}x^6 + \cdots = -2\ln(\cos x)$$

6. Conclusions

In this paper, the exact solutions of nonlinear differential equations are obtained by using Aboodh transform and differential transform methods. This method is more efficient and easy to handle such differential equations in comparison to other methods. the results reveal that this method is very efficient, simple and can be applied to other nonlinear problems.

Appendix

Table A1. *Aboodh transform of some functions.*

$f(t)$	$A[f(t)] = k(v)$
1	$\frac{1}{v^2}$
t	$\frac{1}{v^3}$
e^{at}	$\frac{1}{v^2 - av}$
t^n	$\frac{n!}{v^{n+2}}$
$\sin(at)$	$\frac{a}{v(v^2+a^2)}$
$\cos(at)$	$\frac{1}{(v^2+a^2)}$
$\sinh at$	$\frac{a}{v(v^2 - a^2)}$
$t \cosh at$	$\frac{1}{(v^2 - a^2)}$

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