American Journal of Applied Mathematics
2016; 4(4): 186-191
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20160404.14
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

Solutions of Ordinary Differential System by Using Symmetry Group

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To cite this article:

Sameer Qasim Hasan, Wissam Salman Abd. Solutions of Ordinary Differential System by Using Symmetry Group. *American Journal of Applied Mathematics.* Vol. 4, No. 4, 2016, pp. 186-191. doi: 10.11648/j.ajam.20160404.14

Received: June 19, 2016; **Accepted**: June 28, 2016; **Published**: July 28, 2016

Abstract: In this paper the idea is to use a coordinate transformation that takes a system of ordinary differential equations with no obvious solution to system of integrables. The techniques that which introduced presented the cases of symmetry transformations also the particular case that involving an integrating factory homogeneous coordinates all these transformations are interest in lie groups theory.

Keywords: Coordinate Transformation, System Ordinary Differential Equations, Symmetry Transformation, Lie Groups Theory

1. Introduction

Generally speaking symmetry can be understood as a mapping between mathematical objects that preserves some properties of those objects. The most obvious example are geometric shapes. Consider a square. If we rotate a square by 90 degrees we obtain the exact same square again that is a square with the same orientation. The same is true for rotations of 180, 270, or any integer multiple of 90 degrees. However, if we rotate the square by, say, 60 degrees we obtain an object that is distinguishable from the original its orientation. So the rotations of the square by integer multiples of 90 degrees form a (discrete) set of mapping from the square to itself i.e. a square is symmetric under certain rotations. Now consider a circle. We can also rotate a circle in its plane and obtain the same circle we started with, but now we may rotate by any amount we like and still obtain the same object we started with. Thus the rotations of the circle form a continuous set of symmetries of the circle [1].

Study of symmetry provides one of the most appealing practical applications of group theory. This was extensively shown by sophism Lie in the nineteenth century [1872-1899]. He investigated the continuous group of transformations leaving differential equations invariant creating what is now called the symmetry analysis of differential equations. His aim was to solve non-linear differential equations, which to some extended are may be cumbersome to solve. Intuitively, Lie's method of solving differential equations enables differential equations to be solved in an algebraic approach an as it is put across in [4]. In [5] John Starrett. Solving Differential Equations by Symmetry Groups, and Jacob Harry Burkman. Symmetry group solutions to differential equations [6]. also Peter J. Olver and Philip Rosenau. Group-Invariant solutions of differential equations [7]. and E. S. Cheb-Terrab and T. Kolokolnikov. (2002). First- order ordinary differential equations symmetry and linear transformations [8]. and in [9] Mehmet can. Lie symmetries of differential equations by computer algebra, finally G. W. Bluman and S. Kumei.(1989).Symmetries and differential equations [10].

The solutions for system of ordinary differential equation exhibit symmetries with some properties for solving those solutions so, we state with the following definitions:

Definition 1. [2] A group is a set G together with a group operation such that for any two elements f and g of G, the product $f \cdot g$ is again an element of G.

Definition 2. [3] An r-parameter Lie group is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operation

$$m: G \times G \mapsto G m(f,g) = f \cdot g f, g \epsilon G$$

and the inversion

$$i: G \longmapsto Gi(f) = f^{-1}f\epsilon G$$

are smooth maps between manifolds.

In the study of ODE's and their solutions, we are interested in particular forms. Let $G_\epsilon: (x,y) \longmapsto f((x,y),\epsilon) = (u,v)$ be a transformation from R^2 to R^2 depending on a parameter $\epsilon \in R$
that has the following properties:

1. G_ϵ is bijective
2. $G_a \circ G_b = G_{a+b}$
3. $G_0 = I, the\ identity. That\ is, f((x,y),0) = (x,y).$
4. For each a there exists b such that $G_a \circ G_b = G_0$.
5. The function $f\ is\ C^\infty\ on\ D \subset R^2, (x,y) \in$ D and analytic with respect to ϵ.

Then the set G of transformations G_ϵ is an additive transformation group.

Further, G is a one-parameter Lie group.

Definition 3. [2] Asymmetry of a differential equation is an invertible transformation that maps solutions to solutions

For the purposes of this paper we are interested in one-parameter Lie groups that are the symmetries of a first order ODE. These are transformations of the form $G_\epsilon(x,y) = (u(x,y), v(x,y)$ with anonzero Jacobian $u_x v_y - u_y v_x \neq 0$. If the solutions of the ODE are defined on D ∈ R^2, then the transformation G: D ⟼ D maps solutions to solutions.

*Example 1.*consider the system

$$\frac{dy_1}{dx} = \frac{\beta y_2}{x}$$

$$\frac{dy_2}{dx} = \frac{y_1}{\beta x}$$

$$y_1 = \beta x^\alpha$$

$$y_2 = x^\alpha$$

$$G_\epsilon(x, y_1, y_2) \longmapsto (u(\epsilon), v_1(\epsilon), v_2(\epsilon)) = (x, e^\epsilon y_1, e^\epsilon y_2)$$

$$u(\epsilon) = x, v_1(\epsilon) = e^\epsilon \beta u^\alpha, v_2(\epsilon) = e^\epsilon u^\alpha$$

Definition 4. [2] For a given point (x_0, y_0) and a continuous symmetry $G_\in$ of an ODE, the set of points $\{G_\in(x,y): a < \in < b\} \subset R^2$ is an orbit of $G_\in$

As ∈ varies continuously $G_\in(x_0,y_0)$ traces a continuous curve in R^2 that is transverse to the solution curves of the ODE. Along an orbit, a change in ∈ maps a solution curve to another solution curve.

2. Symmetry of Nonlinear System

The develop a way to find the symmetries of a particular system.

Lemma 1. A differential equation with a translational symmetry in the dependent variable is separable.

proof.

1. Let $G_{\epsilon_1}(u,v,p) = (u_1(\epsilon_1), v_1(\epsilon_1), p_1(\epsilon_1)) = (u, v+\epsilon_1, p+\epsilon_1)$
2. Let $G_{\epsilon_2}(u,v,p) = (u_1(\epsilon_2), v_1(\epsilon_2), p_1(\epsilon_2)) = (u, v+\epsilon_2, p+\epsilon_2)$

be a symmetry of the ODE

$$\frac{dv}{du} = w_1(u,v,p)$$

$$\frac{dp}{du} = w_2(u,v,p)$$

then

$$1.\ \mathrm{w_1(u, v+\epsilon_1, p+\epsilon_1) = w_1(u_1(\epsilon_1), v_1(\epsilon_1), p_1(\epsilon_1))} = \frac{\mathrm{dv_1}}{\mathrm{du_1}} = \frac{\mathrm{d(v+\epsilon_1)}}{\mathrm{du}} = \frac{\mathrm{dv}}{\mathrm{du}} = \mathrm{w_1(u,v,p)}$$

$$2.\ \mathrm{w_2(u, v+\epsilon_2, p+\epsilon_2) = w_2(u_1(\epsilon_2), v_1(\epsilon_2), p_1(\epsilon_2))} = \frac{\mathrm{dp_1}}{\mathrm{du_1}} = \frac{\mathrm{d(p+\epsilon_2)}}{\mathrm{du}} = \frac{\mathrm{dp}}{\mathrm{du}} = \mathrm{w_2(u,v,p)}$$

Thus $w_1, w_2\ are\ independent\ of\ v, p\ and$

$$\frac{dv}{du} = w_1(u)$$

$$\frac{dp}{du} = w_2(u)$$

A separable equation that is integrable

$$v = \int w_1(u)du$$

$$p = \int w_2(u)du$$

They have a way to convert any general symmetry into a translational symmetry, then the integrate to find the solution.

To check whether a transformation is a symmetry, consider two solutions $v(u), p(u)\ and\ v_1(u_1), p_1(u_1)$

That solve the same first order ODE

$$1. \frac{\mathrm{dv}}{\mathrm{du}} = \mathrm{w_1(u,v,p)}$$

$$\frac{dv_1}{du_1} = w_1(u_1, v_1, p_1)$$

$$2. \frac{dp}{du} = w_2(u,v,p)$$

$$\frac{dp_1}{du_1} = w_2(u_1, v_1, p_1)$$

A symmetry of the ODE is a transformation that maps

$$(u,v,p) \longmapsto (u_1(u,v,p), v_1(u,v,p), p_1(u,v,p))$$

If we expand the differentials, and find:

$$w_1(u_1, v_1, p_1) = \frac{dv_1}{du_1} = \frac{v_{1_u}du + v_{1_v}dv}{u_{1_u}du + u_{1_v}dv} = \frac{v_{1_u}du + v_{1_v}v'(u)du}{u_{1_u}du + u_{1_v}v'(u)du} = \frac{v_{1_u} + v_{1_v}w_1(u,v,p)}{u_{1_u} + u_{1_v}w_1(u,v,p)} \quad (1)$$

$$w_2(u_1,v_1,p_1)=\frac{dp_1}{du_1}=\frac{p_{1u}du+p_{1p}dp}{u_{1u}du+u_{1p}dp}=\frac{p_{1u}du+p_{1p}p'(u)du}{u_{1u}du+u_{1p}p'(u)du}=\frac{p_{1u}+p_{1v}w_2(u,v,p)}{u_{1u}+u_{1v}w_2(u,v,p)} \quad (2)$$

In principle we might like to solve this nonlinear partial differential equation for u_1, v_1 *and* p_1 to explicity obtain the symmetry

$(u,v,p)\longmapsto\big(u_1(u,v,p),v_1(u,v,p),p_1(u,v,p)\big)$ but this is generally impossible.

However, we can find a linearization from the Taylor series expansion of (1), (2) from this and can construct the symmetry. This amounts to finding the vector field that is everywhere tangent to the coordinate curves in the new coordinate system (u_1,v_1,p_1)

1. This is accomplished by expanding by expanding

$w_1(u_1,v_1,p_1), u_1, v_1$ *and* p_1 in Taylor series about $\epsilon_1=0$

$$w_1(u_1,v_1,p_1)=w_1(u,v,p)+\epsilon_1(\mathrm{w_u}(u,v,p)\frac{dU_1}{d\epsilon_1}|_{\epsilon_1=0}$$
$$+\epsilon_1(\mathrm{w_v}(u,v,p)\frac{dV_1}{d\epsilon_1}|_{\epsilon_1=0}$$
$$+\epsilon_1(\mathrm{w_p}(\mathrm{u,v,p})\frac{dP_1}{d\epsilon_1}|_{\epsilon_1=0}+\mathrm{o}(\epsilon_1{}^2)$$

$$\mathrm{u_1(u,v,p)}=u+\epsilon_1\left(\frac{\mathrm{dU_1}}{\mathrm{d\epsilon_1}}|_{\epsilon_1=0}\right)+\mathrm{o}(\epsilon_1{}^2)$$

$$\mathrm{v_1(u,v,p)}=v+\epsilon_1\left(\frac{\mathrm{dV_1}}{\mathrm{d\epsilon_1}}|_{\epsilon_1=0}\right)+\mathrm{o}(\epsilon_1{}^2)$$

$$\mathrm{p_1(u,v,p)}=p+\epsilon_1\left(\frac{\mathrm{dp_1}}{\mathrm{d\epsilon_1}}|_{\epsilon_1=0}\right)+\mathrm{o}(\epsilon_1{}^2)$$

2. This is accomplished by expanding by expanding $w_2(u_1,v_1,p_1)u_1, v_1$ *and* p_1 in Taylor series about $\epsilon_2=0$

$$\mathrm{w_2(u_1,v_1,p_1)}=\mathrm{w_2(u,v,p)}+\epsilon_2(\mathrm{w_u(u,v,p)}\frac{\mathrm{dU_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}$$
$$+\epsilon_2(\mathrm{w_v(u,v,p)}\frac{\mathrm{dV_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}$$
$$+\epsilon_2(\mathrm{w_P(u,v,p)}\frac{\mathrm{dP_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}+\mathrm{o}(\epsilon_2{}^2)$$

$$\mathrm{u_1(u,v,p)}=u+\epsilon_2\left(\frac{\mathrm{dU_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}\right)+\mathrm{o}(\epsilon_2{}^2)$$

$$\mathrm{v_1(u,v,p)}=v+\epsilon_2\left(\frac{\mathrm{dV_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}\right)+\mathrm{o}(\epsilon_2{}^2)$$

$$\mathrm{p_1(u,v,p)}=p+\epsilon_2\left(\frac{\mathrm{dP_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}\right)+\mathrm{o}(\epsilon_2{}^2)$$

1. since

$(\mathrm{u_1}(\epsilon_1),\mathrm{v_1}(\epsilon_1),\mathrm{p_1}(\epsilon_1))=\mathrm{G}_{\epsilon_1}(\mathrm{u,v,p})=$
$(\mathrm{U_1}(\mathrm{u,v,p},\epsilon_1),\mathrm{V_1}(\mathrm{u,v,p},\epsilon_1),\mathrm{P_1}((\mathrm{u,v,p},\epsilon_1)$ we have

$$\frac{dU_1}{d\epsilon_1}|_{\epsilon_1=0}=\mathrm{u_1}'(0)=\xi(\mathrm{u,v,p}),\frac{\mathrm{dV_1}}{\mathrm{d\epsilon_1}}|_{\epsilon_1=0}=\mathrm{v_1}'(0)$$
$$=\eta(\mathrm{u,v,p})$$

And $\frac{\mathrm{dP_1}}{\mathrm{d\epsilon_1}}|_{\epsilon_1=0}=\mathrm{p_1}'(0)=\tau(\mathrm{u,v,p})$

The vector fields $(\mathrm{u_1}'(\epsilon_1),\mathrm{v_1}'(\epsilon_1))$ are the slope of the orbits of G_{ϵ_1} and thus can be integrated to obtain the the orbits. If $\xi(\mathrm{u,v,p})$=0 then the orbits of $\mathrm{G}_{\epsilon_1}$are vertical lines. Otherwise they are the solution to

$$\frac{dv}{du}=\frac{\eta(\mathrm{u,v,p})}{\xi(\mathrm{u,v,p})}$$

2. since

$(\mathrm{u_1}(\epsilon_2),\mathrm{v_1}(\epsilon_2),\mathrm{p_1}(\epsilon_2))=\mathrm{G}_{\epsilon_2}(\mathrm{u,v,p})=$
$(\mathrm{U_1}(\mathrm{u,v,p},\epsilon_2),\mathrm{V_1}(\mathrm{u,v,p},\epsilon_2),\mathrm{P_1}((\mathrm{u,v,p},\epsilon_2)$ we have

$$\frac{dU_1}{d\epsilon_2}|_{\epsilon_2=0}=\mathrm{u_1}'(0)=\xi(\mathrm{u,v,p}),\frac{\mathrm{dV_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}=\mathrm{v_1}'(0)$$
$$=\eta(\mathrm{u,v,p})$$

And $\frac{\mathrm{dP_1}}{\mathrm{d\epsilon_2}}|_{\epsilon_2=0}=\mathrm{p_1}'(0)=\tau(\mathrm{u,v,p})$

The vector fields $(\mathrm{u_1}'(\epsilon_2),\mathrm{p_1}'(\epsilon_2))$ are the slope of the orbits of G_{ϵ_2} and thus can be integrated to obtain the the orbits. If $\xi(\mathrm{u,v,p})$=0 then the orbits of G_{ϵ_2} are vertical lines. Otherwise they are the solution

$$\frac{dp}{du}=\frac{\tau(\mathrm{u,v,p})}{\xi(\mathrm{u,v,p})}$$

Definition 5. [3] the canonical coordinates (r(u, v, p)$,s_1(u,v,p),s_2(u,v,p)$) of a differential equation are the coordinates in which the equation becomes separable.

In the simplest case we look for coordinates that admits a symmetry

$s_{\epsilon_{1,2}}:(r,s_1,s_2)\longmapsto(\mathrm{r},s_1+\epsilon_{1,2},s_2+\epsilon_{1,2})$. Then the tangent vector at

$(r(\mathrm{G}_{\epsilon_{1,2}}(u,v,p),s_1(\mathrm{G}_{\epsilon_{1,2}}(u,v,p),s_2(\mathrm{G}_{\epsilon_{1,2}}(u,v,p))=$
$\big(r(u,v,p),s_1(u,v,p),s_2(u,v,p)\big)$ is

$$\left(\frac{dr}{d\epsilon_1}|_{\epsilon_{1,2}=0},\frac{d(s_1+\epsilon_{1,2})}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0},\frac{d(s_2+\epsilon_{1,2})}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}\right)=(0{,}1{,}1)$$

Taking derivatives with respect to ϵ_1 *at* $\epsilon_1=0$, and get

$$\frac{dr(u,v,p)}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}=\frac{dr}{du}\frac{du}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}+\frac{dr}{dv}\frac{dv}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}$$
$$+\frac{dr}{dp}\frac{dp}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}=\frac{dr}{dt}\xi+\frac{dr}{du}\eta+\frac{dr}{dv}\tau$$
$$=0$$

$$\frac{ds_1(u,v,p)}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}$$
$$= \frac{ds_1}{du}\frac{du}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0} + \frac{ds_1}{dv}\frac{dv}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}$$
$$+ \frac{ds_1}{dp}\frac{dp}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0} = \frac{ds_1}{du}\xi + \frac{ds_1}{dv}\eta + \frac{ds_1}{dp}\tau$$
$$= 1$$

$$\frac{ds_2(u,v,p)}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}$$
$$= \frac{ds_2}{du}\frac{du}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0} + \frac{ds_2}{dv}\frac{dv}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}$$
$$+ \frac{ds_2}{dp}\frac{dp}{d\epsilon_{1,2}}|_{\epsilon_{1,2}=0}$$
$$= \frac{ds_2}{du}\xi + \frac{ds_2}{dv}\eta + \frac{ds_2}{dp}\tau = 1$$

or

$$r_u\xi + r_v\eta + r_p\tau = 0 \quad (3)$$

$$s_{1_u}\xi + s_{1_v}\eta + s_{1_p}\tau = 1 \quad (4)$$

$$s_{2_u}\xi + s_{2_v}\eta + s_{2_p}\tau += 1 \quad (5)$$

A system of first order linear partial differential equations that can be solved for $r = r(u,v,p), s_1(u,v,p)$ and $s_2(t,u,v,p)$. In the canonical coordinates equation (1), (2) becomes the separable equation

$$\frac{ds_1}{dr} = \frac{s_{1_u} + s_{1_v}v'}{r_u + r_v v'}$$

$$\frac{ds_2}{dr} = \frac{s_{2_u} + s_{2_p}p'}{r_u + r_p p'}$$

which can be solved for $s_1(r)$ *and* $s_2(r)$. In order to recover the solution in the original (u, v, p) coordinates the transformation$(u,v,p) \longmapsto (r,\ s_1,\ s_2)$ must be invertible i.e

$$r_u s_{1_v} - r_v s_{1_u} \neq 0$$

$$r_u s_{2_p} - r_p s_{2_u} \neq 0$$

*Example 1.*consider the equations

$$\frac{dv}{du} = \frac{1+p}{v} \quad (6)$$

$$\frac{dp}{du} = \frac{u+vu}{p} \quad (7)$$

from (6) we get

$$r = u$$

$$\frac{dv}{du} = \frac{1+p}{v}$$

$$vdv = (1+p)du$$

$$\frac{v^2}{2} = u(1+p) + c$$

$$\therefore\ s_1 = \frac{v^2}{2},\ s_1 = 1 + p$$

$$\frac{ds_1}{dr} = \frac{s_{1_u} + s_{1_v}v'}{r_u + r_v v'} = \frac{0 + vv'}{1+0} = vv'$$

$$\frac{ds_2}{dr} = \frac{s_{2_u} + s_{2_p}p'}{r_u + r_p p'} = \frac{0 + p'}{1+0} = p'$$

now from (7) we obtain

$$r = u$$

$$\frac{dp}{du} = \frac{u+vu}{p}$$

$$pdp = (u+vu)du$$

$$\frac{p^2}{2} = \frac{u^2}{2} + v\frac{u^2}{2} + c$$

$$\frac{p^2}{2} = \frac{u^2}{2}(1+v) + c$$

$$\therefore\ s_1 = 1 + v, s_2 = \frac{p^2}{2}$$

$$\frac{ds_1}{dr} = \frac{s_{1_u} + s_{1_v}v'}{r_u + r_v v'} = \frac{0 + v'}{1+0} = v'$$

$$\frac{ds_2}{dr} = \frac{s_{2_u} + s_{2_p}p'}{r_u + r_p p'} = \frac{0 + pp'}{1+0} = pp'$$

3. Standard Integration Techniques

There is some standard techniques for solving differential equations and see that they are unified by methods of symmetry.

Integrating Factors

For an inhomogeneous first- order linear equation

$$\frac{dv}{du} + F(u)v = G_1(u) \quad (8)$$

$$\frac{dp}{du} + F(u)p = G_2(u) \quad (9)$$

The standard approach is to multiply both side by an integrating factor and integrate to obtain

$$v = e^{\int_0^u F_1 d\tau}\int e^{-\int_0^u F_1 d\tau}\,G_1(u)du$$

$$p = e^{\int_0^u F_2 d\tau}\int e^{-\int_0^u F_2 d\tau}\,G_2(u)du$$

we observe how this solution arises using symmetries the homogeneous equation

$$v_h{}' + F_1(u)v_h = 0$$

$$p_h{}' + F_2(u)p_h = 0$$

are separable and directly integrable

$$v_h = e^{\int_0^u F_1 d\tau}$$

$$p_h = e^{\int_0^u F_2 d\tau}$$

Since equation (8),(9) are linear, if v_p, p_p are a solution to (8),(9)

$$v = v_p + v_h$$

$$p = p_p + p_h$$

are also solution. Hence the transformation

$$G_{\epsilon_1}: (u, v, p) \longmapsto \left(u, v + \epsilon_1 v_h(u), p + \epsilon_1 p_h(u)\right)$$

$$G_{\epsilon_2}: (u, v, p) \longmapsto (u, v + \epsilon_2 v_h(u), p + \epsilon_2 p_h(u))$$

are a symmetry of (9),(10). The orbits of $G_{\epsilon_1}, G_{\epsilon_2}$ are vertical lines, so we can set r=u for one of the canonical coordinates, also have

$$v_1{}'(\epsilon_1) = v_h(u)$$

$$p_1{}'(\epsilon_2) = p_h(u)$$

and get

$$s_1 = \frac{v}{v_h}$$

$$s_2 = \frac{p}{p_h}$$

In the canonical coordinates (r,s_1, s_2), equation (8), (9) becomes:

$$1. \frac{\mathrm{ds_1}}{\mathrm{dr}} = \frac{\mathrm{s_{1_u}\,du + s_{1_v}\,dv}}{\mathrm{r_u\,du + \ r_v\,dv}} = -\frac{\mathrm{v\,v_h{}'(u)}}{\mathrm{v_h(u)^2}} + \frac{1}{\mathrm{v_h(u)}}\frac{\mathrm{dv}}{\mathrm{du}}$$

$$= \frac{vG_1(u)v_h(u)}{v_h(u)^2} + \frac{1}{v_h(u)}\frac{dv}{du}$$

$$= \frac{F_1(u)v}{v_h(u)} + \frac{1}{v_h(u)}(G_1(u) - F_1(u)v)$$

$$= \frac{G_1(u)}{v_h(u)} = \frac{G_1(r)}{v_h(r)}$$

$$= e^{-\int_0^u F_1 d\tau} G_1(u)$$

Integrating obtains

$$s_1(r) = s_1(u) = \int e^{-\int_0^u F_1 d\tau} G_1(u) du$$

and inverting the transformation yields the solution in the original coordinates (u, v, p):

$$v = s_1(u)v_h = e^{-\int_0^u F_1 d\tau} \int e^{-\int_0^u F_1 d\tau} G_1(u)\, du$$

$$2. \frac{ds_2}{dr} = \frac{s_{2_u}du + s_{2_p}dp}{r_u du + \ r_p dp} = -\frac{p p_h{}'(u)}{p_h(u)^2} + \frac{1}{p_h(u)}\frac{dp}{du}$$

$$= \frac{pG_2(u)p_h(u)}{p_h(u)^2} + \frac{1}{p_h(u)}\frac{dp}{du}$$

$$= \frac{F_2(u)p}{p_h(u)} + \frac{1}{p_h(u)}(G_2(u) - F_2(u)p)$$

$$= \frac{G_2(u)}{p_h(u)} = \frac{G_2(r)}{p_h(r)} = e^{-\int_0^u F_2 d\tau} G_2(u)$$

Integrating obtains

$$s_2(r) = s_2(u) = \int e^{-\int_0^u F_2 d\tau} G_2(u) du$$

and inverting the transformation yields the solution in the original coordinates (u, v, p):

$$p = s_2(u)p_h = e^{-\int_0^u F_2 d\tau} \int e^{-\int_0^u F_2 d\tau}(u)\, G_2(u) du$$

4. Homogeneous Coordinates

A homogeneous equation has the form

$$v' = F\left(\frac{v}{u}\right) \qquad (10)$$

$$P' = F\left(\frac{p}{u}\right) \qquad (11)$$

The transformations

$$G_{\epsilon_1}: (u, v, p) \mapsto (e^{\epsilon_1}u, e^{\epsilon_1}v, e^{\epsilon_1}p) = (u_1, v_1, p_1)$$

$$G_{\epsilon_2}: (u, v, p) \mapsto (e^{\epsilon_2}u, e^{\epsilon_2}v, e^{\epsilon_2}p) = (u_1, v_1, p_1)$$

are symmetries of (11),(12) and since

$(u_1{}'(\epsilon_1), v_1{}'(\epsilon_1), p_1{}'(\epsilon_1))$ $=e^{\epsilon_1}(u, v, p)$ the orbite of G_{ϵ_1}

$(u_1{}'(\epsilon_2), v_1{}'(\epsilon_2), p_1{}'(\epsilon_2))$ $=e^{\epsilon_2}(u, v, p)$ the orbite of G_{ϵ_2}

are straight lines through the origin plus the origin itself. Thus

$$r = \frac{v}{u}, r = \frac{p}{u}$$

are constant on the orbits. If we let $s_1 = \ln u$, $s_2 = \ln u$ equation (4), (5) is satisfied and the Jacobian of the transformation $(u, v, p) \longmapsto (r, s_1, s_2)$ is non-zero. In these coordinates (11), (12) is transformation into the separable equation

$$1. \frac{ds_1}{dr} = \frac{s_{1_u} + s_{1_v}v'}{r_u + \ r_v v'} = \frac{\frac{1}{u}}{-\frac{v}{u^2} + \frac{F_1\left(\frac{v}{u}\right)}{u}} = \frac{1}{F_1\,(r) - r}$$

$$v = e^{\int \frac{1}{F\left(\frac{v}{u}\right) - \frac{v}{u}}}$$

$$v = re^{\int \frac{1}{F\left(\frac{v}{u}\right) - \frac{v}{u}}}$$

$$v = re^{\int \frac{1}{F(r)-r}}$$

$$2. \frac{ds_2}{dr} = \frac{s_{2_u} + s_{2_p} p'}{r_u + r_p p'} = \frac{\frac{1}{u}}{-\frac{p}{u^2} + \frac{F_2(\frac{p}{u})}{u}} = \frac{1}{F_2(r) - r}$$

$$p = e^{\int \frac{1}{F(\frac{p}{u}) - \frac{p}{u}}}$$

$$p = re^{\int \frac{1}{F(\frac{p}{u}) - \frac{p}{u}}}$$

$$p = re^{\int \frac{1}{F(r)-r}}$$

5. Conclusion

In this paper, we concluded that the symmetry transform for solving system of ordinary differential equation it is obtain by some technical was give with some system of integrables.

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