

On the solution of a boundary value problem related to the heat transmission

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To cite this article:

Khanlar R. Mamedov, Volkan Ala. On the Solution of a Boundary Value Problem related to the Heat Transmission. *American Journal of Applied Mathematics*. Vol. 2, No. 2, 2014, pp. 54-59. doi: 10.11648/j.ajam.20140202.12

Abstract: In this study, we consider a heat transmission problem which has derivative with respect to the time in boundary condition. Applying the separation of variables method, we get a Sturm-Liouville equation with discontinuous coefficient and a spectral parameter dependent boundary condition. For this spectral problem, the operator theoretic formula is given, the resolvent operator constructed and the expansion formula with respect to the eigenfunctions obtained. Using the expansion formula, the solution of the heat problem expressed.

Keywords: Sturm-Liouville Operator, Resolvent Operator, Expansion Formula

1. Introduction

In this paper, we consider the following heat transmission problem:

$$\rho(x) \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - q(x)v, \quad x > 0, \quad t > 0, \quad (1)$$

$$\left(-\beta_0 v + \frac{\partial v}{\partial x} + \beta_1 \frac{\partial v}{\partial t} \right) \Big|_{x=0} = 0, \quad t > 0, \quad (2)$$

$$v(x, 0) = \Phi_0(x), \quad x > 0, \quad (3)$$

where

$$v(0, 0) = v_0^*. \quad (4)$$

In keeping with the physics of the problem, we assume that the boundary condition at $x = \infty$, the solution $v(x, t)$ remains bounded:

$$\lim_{x \rightarrow \infty} |v(x, t)| < \infty.$$

$q(x)$ is a real-valued function integrable on $[0, +\infty)$ satisfying the following condition:

$$\int_0^{\infty} x |q(x)| dx < \infty.$$

$\beta_0, \beta_1 < 0$ are real numbers, $\rho(x)$ is the following piecewise-constant function which has discontinuity at point $a \in [0, +\infty)$.

$$\rho(x) = \begin{cases} \alpha^2, & 0 \leq x < a, \\ 1, & x \geq a. \end{cases}$$

with $0 \leq \alpha \neq 1$.

We assume that $\Phi_0(x)$ is sufficiently smooth continuous function which has first and second continuous derivatives for $x > 0$. Suppose that, $\Phi_0''(x) = O(x^{-2})$ as $x \rightarrow \infty$, $x > 0$. $\Phi_0(x) = O(x^{-1})$ as $x \rightarrow \infty$.

Problem (1)–(4) was researched for the case of $\rho(x) = 1$, $q(x) = 0$, $\beta_0 = 0$, $\beta_1 = 1$ by Cohen [1].

A semi-infinite right cylindrical solid with cross section of arbitrary shape and size and with plane terminal face at $x = 0$ has its lateral surface insulated against the heat conduction and has an initial temperature distribution depending on only the longitudinal coordinate x .

A liquid of arbitrary initial temperature is kept well stirred to insure at each instant a uniform temperature throughout it, at time $t = 0$ the plane terminal face of the solid is placed in contact with the liquid. For simplicity it is assumed that the liquid is not accessible to a transfer of heat from its surrounding medium. The problem is to determine the temperature of the liquid and the distribution of temperature in the solid at any time $t > 0$.

When the separation of variables method is used for problem (1),(2) a spectral problem is obtained which involves eigenvalue in boundary condition. Generally, with these type of boundary conditions are encountered when the boundary conditions of the main problem contain partial derivative with respect to the time.

Applying the method of separation of variables to problem (1),(2) we get the spectral problem below:

$$-y'' + q(x)y = \lambda^2 \rho(x)y, \quad (5)$$

$$y'(0) - (\beta_0 + \beta_1 \lambda^2) y(0) = 0, \quad (6)$$

where

$$\lim_{x \rightarrow \infty} |y(x)| < \infty.$$

The spectral analysis of (5) was investigated for finite interval in [2] and for infinite interval in [3]. Expansion formula for Sturm-Liouville problem is given in [4-9] and etc. Direct scattering problem of (5),(6) for the case $\beta_0 = 0$ and $\beta_1 = 1$ was examined in [10]. The inverse problem of scattering theory with discontinuous coefficient for (5) was studied in [12-14].

This paper is organised as follows: In Section 1, we give the integral representation for a second order differential equation with a discontinuous coefficient and the spectral properties of boundary value problem. In Section 2, the operator theoretic formulation of problem (5),(6) in the Hilbert space is defined. In Section 3, the resolvent operator is constructed. In Section 4, the expansion formula with respect to the eigenfunctions is obtained and finally in Section 5, we get formula for the solution of the heat transmission problem (1)-(4) using the expansion formula with respect to the eigenfunctions.

It is known from [14] that for all λ from the closed upper half-plane, (5) has a unique Jost solution $f(x, \lambda)$ which can be represented in the form below:

$$f(x, \lambda) = f_0(x, \lambda) + \int_{\mu^+(x)}^{\infty} K(x, t) e^{i\lambda t} dt, \quad (7)$$

where $\mu^+(x) = x\sqrt{\rho(x)} + a(1 - \sqrt{\rho(x)})$ and the kernel $K(x, t)$ satisfies the inequality,

$$\int_{\mu^+(x)}^{\infty} |K(x, t)| dt \leq c \int_x^{\infty} |q(t)| dt, \quad 0 < c = \text{const}. \quad (8)$$

For real $\lambda \neq 0$, the functions $f(x, \lambda)$ and $\overline{f(x, \lambda)}$ form the fundamental system of solutions of equation (5) and the Wronskian of this system is equal to $2i\lambda$:

$$W\{f(x, \lambda), \overline{f(x, \lambda)}\} = f'(x, \lambda) \overline{f(x, \lambda)} - f(x, \lambda) \overline{f'(x, \lambda)} = 2i\lambda.$$

Let be $\omega(x, \lambda)$ the solution of (5) satisfying the initial data

$$\omega(0, \lambda) = 1, \quad \omega'(0, \lambda) = \beta_0 + \beta_1 \lambda^2.$$

We define

$$\varphi(\lambda) := f'(0, \lambda) - (\beta_0 + \beta_1 \lambda^2) f(0, \lambda).$$

Using the form (7) for the solution of (5),(6) it can be proved that $\varphi(\lambda)$ may have only a finite number of zeros in the half plane $\text{Im } \lambda > 0$. Moreover, all these zeros are simple and lie on the imaginary axis. These propositions are proved by the same method that was used in [7] (Lemma 3.1.6).

(Also see [11], Lemma 3 and [13], Lemma 2.)

Let us denote differentiation with respect to λ with a dot and let $\lambda = i\lambda_j$ ($\lambda_j > 0$) be the zeros of $\varphi(\lambda)$ in the half plane $\text{Im } \lambda > 0$. For $\lambda = i\lambda_j$, $j = 1, 2, \dots, n$, we get

$$\begin{aligned} m_j^{-2} &= \int_0^{\infty} \rho(x) |f(x, i\lambda_j)|^2 dx + \beta_1 |f(0, i\lambda_j)|^2 \\ &= -\frac{1}{2i\lambda_j} \dot{\varphi}(i\lambda_j) f(0, i\lambda_j). \end{aligned} \quad (9)$$

The numbers m_j^{-2} are called the *normalizing numbers* of problem (5),(6). (see [13], formula (1.14))

2. The Operator Theoretic Formulation

In the Hilbert Space $H_\rho = L_{2,\rho}(0, \infty) \times \square$ an inner product is defined by

$$\langle F, G \rangle := \int_0^{\infty} F_1(x) \overline{G_1(x)} \rho(x) dx - \frac{1}{\beta_1} F_2 \overline{G_2}$$

where

$$F = \begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix}, \quad G = \begin{pmatrix} G_1(x) \\ G_2 \end{pmatrix} \in H_\rho.$$

For convenience, we put

$$R_0[y] = \beta_0 y(0) - y'(0),$$

$$R'_0[y] = \beta_1 y(0).$$

Let us define operator (see [1].)

$$L(F) := \begin{pmatrix} l(F_1) \\ F'_1(0) \end{pmatrix}.$$

with domain

$$D(L) := \left\{ F \in H_\rho \mid F_1(x), F'_1(x) \text{ are AC}[0, b] \subset [0, \infty), \right. \\ \left. l(F_1) \in L_{2,\rho}(0, \infty), F_2 = \beta_1 F_1(0). \right\}$$

(AC: absolutely continuous)

where

$$l(F_1) = \frac{1}{\rho(x)} \{-F_1'' + q(x)F_1\}.$$

Problem (5),(6) is equivalent to the equation $Ly = \lambda^2 y$ and the operator L with the domain $D(L)$ is self-adjoint in the Hilbert space of H_ρ .

3. The Resolvent Operator

We suppose that λ^2 is not a spectrum point of operator L , then the resolvent $R_{\lambda^2} = (L - \lambda^2 I)^{-1}$ exists. Now, let us find this expression of the operator $R_{\lambda^2}(L)$.

Theorem 1: *All numbers of the form λ^2 , $\text{Im } \lambda > 0$ $\varphi(\lambda) \neq 0$ belong to the resolvent set of the operator L . The resolvent R_{λ^2} is the integral operator with the kernel which has the following form:*

$$R_{\lambda^2}(L) = \int_0^\infty G(x, \xi; \lambda) F(\xi) d\xi, \quad (10)$$

$$G(x, \xi; \lambda) = \frac{-1}{\varphi(\lambda)} \begin{cases} \omega(x, \lambda) f(\xi, \lambda), & x \leq \xi \leq \infty, \\ f(x, \lambda) \omega(\xi, \lambda), & 0 \leq \xi < x. \end{cases}$$

Proof: Let $F \in D(L)$ and $F = \begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix}$ be zero in

exterior of every finite interval $[0, b] \subset [0, \infty)$. To construct the resolvent operator of L , we need to solve the initial value problem

$$-y'' + q(x)y = \lambda^2 \rho(x)y + \rho(x)F_1(x), \quad (11)$$

$$y'(0) = (\beta_0 + \beta_1 \lambda^2)y(0) + F_2. \quad (12)$$

By applying the variation of constants, we want to find the solution of problem (11),(12) which has a form

$$y(x, \lambda) = c_1(x, \lambda)\omega(x, \lambda) + c_2(x, \lambda)f(x, \lambda), \quad (13)$$

where $\omega(x, \lambda)$ and $f(x, \lambda)$ are solutions of homogenous problem for $\text{Im } \lambda > 0$.

Hence, we get

$$\begin{cases} c_1'(x, \lambda)\omega(x, \lambda) + c_2'(x, \lambda)f(x, \lambda) = 0, \\ c_1'(x, \lambda)\omega'(x, \lambda) + c_2'(x, \lambda)f'(x, \lambda) = -\rho(x)F_1(x). \end{cases} \quad (14)$$

Since $y(x, \lambda) \in L_{2,\rho}(0, \infty)$, then $c_1(\infty, \lambda) = 0$.

Using this relation and (14), we obtain

$$\begin{cases} c_1(x, \lambda) = -\int_x^\infty \frac{f(\xi, \lambda)}{\varphi(\lambda)} F_1(\xi) \rho(\xi) d\xi, \\ c_2(x, \lambda) = c_2(0, \lambda) - \int_0^x \frac{\omega(\xi, \lambda)}{\varphi(\lambda)} F_1(\xi) \rho(\xi) d\xi. \end{cases} \quad (15)$$

Substituting (15) into (13), we obtain

$$y(x, \lambda) = \int_0^\infty G(x, \xi; \lambda) F_1(\xi) \rho(\xi) d\xi + c_2(0, \lambda) f(x, \lambda).$$

Taking the condition (12), we find $c_2(0, \lambda) = \frac{F_2}{\varphi(\lambda)}$.

Thus,

$$y(x, \lambda) = \int_0^\infty G(x, \xi; \lambda) F_1(\xi) \rho(\xi) d\xi + \frac{F_2}{\varphi(\lambda)} f(x, \lambda).$$

Theorem 1 is proved.

Lemma 1: *Let the function $F_1(x)$ is twice continuously differentiable and finite at infinity. Then as $|\lambda| \rightarrow \infty$, $\text{Im } \lambda \geq 0$ the following is valid:*

$$\begin{aligned} \langle G, F \rangle &= \int_0^\infty G(x, \xi; \lambda) F_1(\xi) \rho(\xi) d\xi + \frac{F_2}{\varphi(\lambda)} f(x, \lambda) \\ &= -\frac{F_1(x)}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right). \end{aligned} \quad (16)$$

Proof: Using Theorem 1 and integrating by parts, we write

$$\begin{aligned} \int_0^\infty G(x, \xi; \lambda) F_1(\xi) \rho(\xi) d\xi &= -\int_x^\infty \frac{\omega(x, \lambda) f(\xi, \lambda)}{\varphi(\lambda)} F_1(\xi) \rho(\xi) d\xi \\ &\quad - \int_0^x \frac{\omega(\xi, \lambda) f(x, \lambda)}{\varphi(\lambda)} F_1(\xi) \rho(\xi) d\xi \\ &= \frac{1}{\lambda^2} \int_x^\infty \frac{\omega(x, \lambda)}{\varphi(\lambda)} \{f''(\xi, \lambda) - q(\xi)f(\xi, \lambda)\} F_1(\xi) d\xi \\ &\quad + \frac{1}{\lambda^2} \int_0^x \frac{f(x, \lambda)}{\varphi(\lambda)} \{\omega''(\xi, \lambda) - q(\xi)\omega(\xi, \lambda)\} F_1(\xi) d\xi \\ &= \frac{W\{\omega(x, \lambda), f(x, \lambda)\}}{\lambda^2 \varphi(\lambda)} F_1(x) + \frac{1}{\lambda^2} F_1'(0) f(x, \lambda) \\ &\quad - \frac{1}{\lambda^2 \varphi(\lambda)} \int_0^\infty G(x, \xi; \lambda) \overline{F_1}(\xi) d\xi. \\ \overline{F_1}(\xi) &= -F_1''(\xi) + q(\xi)F_1(\xi). \end{aligned}$$

Therefore,

$$\int_0^\infty G(x, \xi; \lambda) F_1(\xi) \rho(\xi) d\xi + \frac{F_2}{\varphi(\lambda)} f(x, \lambda) =$$

$$-\frac{F_1(x)}{\lambda^2} + \frac{1}{\lambda^2} \left[\frac{F_1'(0)}{\varphi(\lambda)} f(x, \lambda) - \int_0^\infty G(x, \xi; \lambda) \overline{F_1}(\xi) d\xi \right].$$

Lemma 1 is proved.

4. The Expansion Formula

In this section, we obtain the expansion formula by applying the method of Titchmarsh.

$$F(x, \lambda) = \begin{cases} (G_{x, \lambda}, F), & \text{Im } \lambda \geq 0, \\ (G_{x, \bar{\lambda}}, F), & \text{Im } \lambda \leq 0. \end{cases}$$

Let us Γ_R denote the circle of radius R and center is zero which boundary contour is positive oriented. Let us $\Gamma_{R, \varepsilon}$, denote boundary contour positive oriented in plane

$$D = \{z \mid |z| \leq R, |\text{Im } z| \geq \varepsilon\}.$$

$\Gamma'_{R, \varepsilon}$ denotes boundary contour negative oriented in the plane $D = \{z \mid |z| \leq R, |\text{Im } z| \leq \varepsilon\}$. Then, we can use the properties of the integration as follows:

$$\int_{\Gamma_{R, \varepsilon}} = \int_{\Gamma_R} + \int_{\Gamma'_{R, \varepsilon}} \quad (17)$$

Now, multiplying both sides of equality (16) by $\frac{1}{2\pi i} \lambda$ and integrating over λ the contour $\Gamma_{R, \varepsilon}$, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_{R, \varepsilon}} \lambda F(x, \lambda) d\lambda = -\frac{1}{2\pi i} \int_{\Gamma_{R, \varepsilon}} \frac{F_1(x)}{\lambda} d\lambda + \frac{1}{2\pi i} \int_{\Gamma_{R, \varepsilon}} O\left(\frac{1}{\lambda}\right) d\lambda. \quad (18)$$

According to the equation (17), we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_{R, \varepsilon}} \lambda F(x, \lambda) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_R} \lambda F(x, \lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma'_{R, \varepsilon}} \lambda F(x, \lambda) d\lambda. \quad (19)$$

Now let us calculate the integral on the right hand side of (18),

$$\frac{1}{2\pi i} \int_{\Gamma_R} \lambda F(x, \lambda) d\lambda = -\frac{1}{2\pi i} \int_{\Gamma_R} \frac{F_1(x)}{\lambda} d\lambda + \frac{1}{2\pi i} \int_{\Gamma_R} O\left(\frac{1}{\lambda}\right) d\lambda \xrightarrow{R \rightarrow \infty} -F_1(x). \quad (20)$$

Taking (19) into consideration, we get

$$\lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \frac{1}{2\pi i} \int_{\Gamma_{R, \varepsilon}} \lambda F(x, \lambda) d\lambda = -F_1(x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \lambda [F(x, \lambda + i0) - F(x, \lambda - i0)] d\lambda. \quad (21)$$

On the other hand, using the residue calculus, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_{R, \varepsilon}} \lambda F(x, \lambda) d\lambda = \sum_{j=1}^n \text{Res}_{\lambda=i\lambda_j} [\lambda F(x, \lambda)] + \sum_{j=1}^n \text{Res}_{\lambda=-i\lambda_j} [\lambda F(x, \lambda)]. \quad (22)$$

From (21) and (22), we obtain

$$F_1(x) = -\sum_{j=1}^n \text{Res}_{\lambda=i\lambda_j} [\lambda F(x, \lambda)] - \sum_{j=1}^n \text{Res}_{\lambda=-i\lambda_j} [\lambda F(x, \lambda)] + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \lambda [F(x, \lambda + i0) - F(x, \lambda - i0)] d\lambda. \quad (23)$$

Let $\psi(x, \lambda)$ be the solution of (5) satisfying the initial conditions

$$\psi(0, \lambda) = 0, \quad \psi'(0, \lambda) = 1.$$

It is clear that $W[\omega(x, \lambda), \psi(x, \lambda)] = 1$.

Then, for $\text{Im } \lambda > 0$, $f(x, \lambda)$ is written as the linear combination with the solutions $\omega(x, \lambda)$ and $\psi(x, \lambda)$, i.e.

$$f(x, \lambda) = c_1 \omega(x, \lambda) + c_2 \psi(x, \lambda),$$

where

$$c_1 = f(0, \lambda), \quad c_2 = -\varphi(\lambda).$$

Hence,

$$f(x, \lambda) = f(0, \lambda) \omega(x, \lambda) - \varphi(\lambda) \psi(x, \lambda).$$

Consequently, it is clear that

$$G(x, \xi; \lambda) = -\frac{1}{\varphi(\lambda)} \begin{cases} \omega(x, \lambda) [f(0, \lambda) \omega(\xi, \lambda) - \varphi(\lambda) \psi(\xi, \lambda)], & x \leq \xi < \infty, \\ [f(0, \lambda) \omega(x, \lambda) - \varphi(\lambda) \psi(x, \lambda)] \omega(\xi, \lambda), & 0 \leq \xi \leq x. \end{cases}$$

$$= -\frac{f(0, \lambda)}{\varphi(\lambda)} \omega(x, \lambda) \omega(\xi, \lambda) - \begin{cases} \omega(x, \lambda) \psi(\xi, \lambda), & x \leq \xi < \infty, \\ \psi(x, \lambda) \omega(\xi, \lambda), & 0 \leq \xi \leq x. \end{cases}$$

$$F(x, \lambda) = -\frac{1}{\varphi(\lambda)} f(0, \lambda) \omega(x, \lambda) \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi - \psi(x, \lambda) \int_0^x \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi - \omega(x, \lambda) \int_x^\infty \psi(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi + \frac{F_2}{\varphi(\lambda)} f(0, \lambda) \omega(x, \lambda) - F_2 \psi(x, \lambda). \quad (24)$$

It follows that

$$\text{Res}_{\lambda=i\lambda_j} [\lambda F(x, \lambda)] = -\frac{i\lambda_j}{\varphi(i\lambda_j)} f(0, i\lambda_j) \omega(x, i\lambda_j) \int_0^\infty \omega(\xi, i\lambda_j) F_1(\xi) d\xi + \frac{i\lambda_j F_2}{\varphi(i\lambda_j)} f(0, i\lambda_j) \omega(x, i\lambda_j), \quad j = 1, 2, \dots, n.$$

Using properties of $f(x, \lambda)$ and $\omega(x, \lambda)$, we get

$$f(x, \lambda_j) = f(0, \lambda_j) \omega(x, \lambda_j), \quad j = 1, 2, \dots, n. \quad (25)$$

From (9) and (25) we obtain

$$\begin{aligned}
& \operatorname{Re}_s \left[\lambda F(x, \lambda) \right] + \operatorname{Re}_s \left[\lambda F(x, \lambda) \right] \\
& = -m_j^2 f(x, i\lambda_j) \int_0^\infty f(\xi, i\lambda_j) F_1(\xi) d\xi + m_j^2 F_2 f(0, i\lambda_j) f(x, i\lambda_j) \\
& = -m_j (F, U_j(x)) f(x, i\lambda_j), \quad (26)
\end{aligned}$$

where

$$U_j(x) = m_j \begin{pmatrix} f(x, i\lambda_j) \\ f(0, i\lambda_j) \end{pmatrix}, \quad F(x) = \begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix}, \quad j = 1, 2, \dots, n.$$

Now, let us calculate the integral

$$\frac{1}{2\pi i} \int_{-\infty}^\infty \lambda [F(x, \lambda + i0) - F(x, \lambda - i0)] d\lambda.$$

From formula (24) and equality $F(x, \lambda - i0) = \overline{F(x, \lambda + i0)}$, we get

$$\begin{aligned}
& F(x, \lambda + i0) - F(x, \lambda - i0) = \\
& \left[\frac{1}{\varphi(\lambda)} \overline{f(0, \lambda)} - \frac{1}{\varphi(\lambda)} f(0, \lambda) \right] \omega(x, \lambda) \times \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi \\
& = \frac{2i\lambda}{|\varphi(\lambda)|^2} \omega(x, \lambda) \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi - \frac{2i\lambda F_2}{|\varphi(\lambda)|^2} \omega(x, \lambda).
\end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-\infty}^\infty \lambda [F(x, \lambda + i0) - F(x, \lambda - i0)] d\lambda = \\
& \frac{2}{\pi} \int_0^\infty \frac{\lambda^2}{|\varphi(\lambda)|^2} \omega(x, \lambda) \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi - \frac{2F_2}{\pi} \int_0^\infty \frac{\lambda^2 \omega(x, \lambda)}{|\varphi(\lambda)|^2} d\lambda. \quad (27)
\end{aligned}$$

Taking (26), (27) into (23), we obtain the expansion formula according to the eigenfunctions:

$$\begin{aligned}
F_1(x) &= \sum_{j=1}^n m_j (F, U_j(x)) f(x, i\lambda_j) \\
&+ \frac{2}{\pi} \int_0^\infty \frac{\lambda^2 d\lambda}{|\varphi(\lambda)|^2} \omega(x, \lambda) \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi d\lambda \\
&- \frac{2F_2}{\pi} \int_0^\infty \frac{\lambda^2 \omega(x, \lambda)}{|\varphi(\lambda)|^2} d\lambda. \quad (28)
\end{aligned}$$

$$\begin{aligned}
F_2 &= \beta_1 F_1(0) = \sum_{j=1}^n m_j \beta_1 (F, U_j(0)) f(0, i\lambda_j) \\
&+ \frac{2\beta_1}{\pi} \int_0^\infty \frac{\lambda^2 d\lambda}{|\varphi(\lambda)|^2} \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi d\lambda \\
&- \frac{2\beta_1 F_2}{\pi} \int_0^\infty \frac{\lambda^2}{|\varphi(\lambda)|^2} d\lambda. \quad (29)
\end{aligned}$$

5. Solution of the Problem (1)-(4)

In this section, we obtain a formula for the solution of the heat transmission problem using the expansion formula with respect to the eigenfunctions. Making use of (28) and (29), we find the solution of (1)–(2) boundary value problem as the following:

$$\begin{pmatrix} v(x, t) \\ v(0, t) \end{pmatrix} = \begin{pmatrix} F_1(x) \\ F_2 \end{pmatrix} e^{-\lambda^2 t}$$

$$v(x, t) = \sum_{j=1}^n m_j e^{-\lambda_j^2 t} (F, U_j(x)) f(x, i\lambda_j) + \frac{2}{\pi} e^{-\lambda^2 t} \int_0^\infty \frac{\lambda^2 d\lambda}{|\varphi(\lambda)|^2} \omega(x, \lambda) \times$$

$$\int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi d\lambda - \frac{2e^{-\lambda^2 t} F_2}{\pi} \int_0^\infty \frac{\lambda^2 \omega(x, \lambda)}{|\varphi(\lambda)|^2} d\lambda \quad (30)$$

Applying the boundary condition (3) to (30) we obtain

$$\begin{aligned}
v(x, 0) &= \Phi_0(x) = \sum_{j=1}^n m_j (F, U_j(x)) f(x, i\lambda_j) \\
&+ \frac{2}{\pi} \int_0^\infty \frac{\lambda^2 d\lambda}{|\varphi(\lambda)|^2} \omega(x, \lambda) \times \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi d\lambda \\
&- \frac{2F_2}{\pi} \int_0^\infty \frac{\lambda^2 \omega(x, \lambda)}{|\varphi(\lambda)|^2} d\lambda.
\end{aligned}$$

where

$$\begin{aligned}
v(0, 0) &= v_0 = \sum_{j=1}^n m_j (F, U_j(0)) f(0, i\lambda_j) \\
&+ \frac{2\beta_1}{\pi} \int_0^\infty \frac{\lambda^2 d\lambda}{|\varphi(\lambda)|^2} \int_0^\infty \omega(\xi, \lambda) F_1(\xi) \rho(\xi) d\xi d\lambda \\
&- \frac{2\beta_1 F_2}{\pi} \int_0^\infty \frac{\lambda^2}{|\varphi(\lambda)|^2} d\lambda.
\end{aligned}$$

It can be shown that the right side of (30) is absolutely and uniformly convergent on every compact interval $[0, b] \subset [0, \infty)$.

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