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A Convergence Analysis of Discontinuous Collocation Method for IAEs of Index 1 Using the Concept "Strongly Equivalent"

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Abstract: We introduce the concept "strongly equivalent" for integral algebraic equations (IAEs). This definition and its corresponding theorems construct powerful tools for the classifying and analyzing of IAEs (especially numerical analysis). The related theorems with short proofs provide powerful techniques for the complete convergence analysis of discretised collocation methods on discontinuous piecewise polynomial spaces.

Keywords: Volterra Integral Equation, Volterra Equation, Integral equation, Discontinuous Piecewise Polynomial Spaces, Collocation Methods

1. Introduction

Sometimes another solution for a problem may help us to progress more and to dig deeper in science. Integral algebraic equations (IAEs) are mixed system of the first kind and the second kind Volterra integral equations. They are classified by index definition. Recently, their numerical solution using collocation methods on piecewise polynomial spaces has attracted more attention to the researchers. However, there are many questions, unsolved on this subject. The convergence analysis of continuous or discontinuous collocation method for a very restrictive cases like Hessenberg type or low index IAEs has been done more recently (see for example [7,10] for IAEs of index 1, [10] for IAEs of index 2, [11] for IAEs of index 3 and [12,13] for Hessenberg type IAEs of arbitrary index).

Convergence analysis for IAEs of index 1, using continuous collocation methods on piecewise polynomial spaces has not been provided yet. However, a direct complete analysis of linear IAEs of index 1 using discontinuous collocation methods on piecewise polynomial spaces has been done by H. Liang and H. Brunner [10]. The aim of this paper is to get another proof for the convergence analysis for IAEs of index 1 by separating the problem to simple cases. The lemmas and theorems introduced here can help us to obtain convergence analysis of higher index IAEs and more complex methods like continuous collocation methods.

Consider integral algebraic operator of the form

$$\Gamma[A,K,f](y)=A(t)y(t)+\int_0^t K(t,s,y(s))ds-f(t), \qquad (1)$$

on $t\in\mathbb{I}:=[0,T]$, where $A\in\mathbf{C}(\mathbb{I},\mathbb{R}^{r\times r})$ is a singular matrix with constant rank for all $t\in\mathbb{I}$, $f\in\mathbf{C}(\mathbb{I},\mathbb{R}^r)$, $y\in\mathbf{C}(\mathbb{I},\mathbb{R}^r)$, and $K\in\mathbf{C}(\mathbb{D}\times\mathbb{R}^r,\mathbb{R}^r)$ with $\mathbb{D}:=\{(t,s):0\le s\le t\le T\}$. We study Integral Algebraic Equations (IAEs) of the form

$$\Gamma[A,K,f](y)\equiv 0 \qquad (2)$$

where y is the unknown vector. If $K(t,s,y)=k(t,s)y$, where $k\in\mathbf{C}(\mathbb{D},\mathbb{R}^r)$, then, the system (1) is a linear IAE.

The notion of the index is used to classify IAEs. There are different notions of index for classification of IAEs. Gear introduced differential index for IAEs [4]. The left index for system (1) is another notion that was introduced by Russian mathematicians [2, 3]. Lamm [9] introduced "v-smoothing" for the first kind Volterra integral equations which is equivalent with differential index. The tractable index is defined by [1, 6, 10, 11]. In this paper we use "rank-degree" index [12, 13].

Here, we will introduce the concept "Strongly equivalent", IAEs. We will establish theorems on the numerical and analytical solutions of the strongly equivalent IAEs, which reduce the convergence analysis of IAEs. This is done by decomposing the problem to the simple classes. For IAEs of index 1, we divide the system into two famous class of IAEs: A system of the first kind Volterra integral equations and a system of IAEs which was investigated in [7].

The next sections are organized as follows:

In section 2, we recall "rank-degree" index and the conditions under which the system (1) -(2) has unique solution. In section 3, we introduce the concept of "strongly equivalent" for IAEs and we show that the strongly equivalent systems have same solutions. In section 4, we recall discretised collocation methods on discontinuous piecewise polynomial spaces and we show that the approximate solutions of a stated methods for strongly equivalent IAEs are of the same order. In section 5, we divide IAEs of index 1, with regard to the "strongly equivalent" concept, to two categories. Then, theorems about the existence of a unique numerical solution are stated. In section 6, a global convergence analysis of the discretised discontinuous collocation methods (DDCM) solutions is investigated. In section 7, we study the nonlinear systems of IAEs.

2. Index Definition and the Existence of Unique Solutions

Definition 2.1 The matrix $A^{-}(t)$ is called semi-inverse matrix for $A(t)$ if it satisfies

$$A(t)A^{-}(t)A(t) = A(t),$$

which can be rewritten as

$$V(t)A(t) = 0,$$

with

$$V(t) = \mathbf{I} - A(t)A^{-}(t) \tag{3}$$

where $\mathbf{I}$ is an identity matrix.

The following conditions are necessary and sufficient for the existence of a semi-inverse matrix $A^{-}(t)$ with elements in $\mathbf{C}^{p}([0,1],\mathbb{R}^{r\times r})$ [3]:

1. The elements of $A(t)$ belong to $\mathbf{C}^{p}([0,1],\mathbb{R}^{r\times r})$.
2. $rankA(t) = const, \quad \forall t \in [0,1]$.

The IAEs are classified using index notion. From many definitions of index (i. e. [1, 2, 3, 4]), we use the following one.

Definition 2.2 [12, 13] Suppose $A \in \mathbf{C}(\mathbb{I},\mathbb{R}^{r\times r})$ and $K \in \mathbf{C}(\mathbb{D},\mathbb{R}^{r\times r})$. Let

$$A_0 \equiv A, \quad K_0 \equiv k,$$

$$\Lambda_i y = \frac{d}{dt}\Big((\mathbf{I} - A_i(t)A_i^{-}(t))y\Big) + y,$$

$$A_{i+1} \equiv A_i + (\mathbf{I} - A_i(t)A_i^{-}(t))K_i(t,t), \;\; K_{i+1} = \Lambda_i K_i, \; i = 0,\dots,\nu-1.$$

Then, we say that the "rank degree" index of (A,K) is ν if

$$A_i(t) \in \mathbf{C}^1(\mathbb{I},\mathbb{R}^{r\times r}) \quad \text{for } i = 1,\dots,\nu,$$

$$rankA_i(t) = const, \quad \forall t \in \mathbb{I} \;\text{ for } i = 0,\dots,\nu,$$

$$\det A_i = 0, \text{ for } i = 0,\dots,\nu-1, \; \det A_\nu \neq 0.$$

Moreover, we say that the "rank-degree" index of linear system (2) is ν ($ind_r = \nu$) if in addition to the above hypotheses, we have

$$F_0 \equiv f, \; F_{i+1} \equiv \Lambda_i F_i, \; i = 0,\dots,\nu-1,$$

$$F_i \in \mathbf{C}^1(\mathbb{I},\mathbb{R}^{r}), \; i = 1,\dots,\nu,$$

where $\mathbf{I}$ is an identity operator.

Theorem 2.1 [12, 13] Suppose the following conditions are satisfied for (2):

1. $ind_r = \nu \geq 1$,
2. $A_0(t) \in \mathbf{C}(\mathbb{I},\mathbb{R}^{r\times r})$, $F_0(t) \in \mathbf{C}(\mathbb{I},\mathbb{R}^{r})$, $K_0 \in \mathbf{C}(\mathbb{D},\mathbb{R}^{r\times r})$, $A_i(t) \in \mathbf{C}^1(\mathbb{I},\mathbb{R}^{r\times r})$, $F_i(t) \in \mathbf{C}^1(\mathbb{I},\mathbb{R}^{r})$, $K_i \in \mathbf{C}^1(\mathbb{D},\mathbb{R}^{r\times r})$, for $i = 1,\cdots,\nu$,
3. $A_i(0)A_\nu^{-1}(0)F_\nu(0) = F_i(0)$ for $i = 0,\cdots\nu-1$, (consistency conditions)
4. $rank(\mathbf{I} - A_iA_i^{-}) = \deg\det\Big(\lambda(\mathbf{I} - A_iA_i^{-}) + (\mathbf{I} - A_iA_i^{-})' + \mathbf{I}\Big) = c$,

Then the system (2) has a unique solution on $\mathbb{I}$.

The condition 2 of the Theorem 2.1 will not be used in the next sections, since the definition of index includes this condition.

3. Strongly Equivalent

In this section, we introduce the concept of "strongly equivalent" systems.

Definition 3.1 Two systems $\Gamma[A,K,f](y) \equiv 0$ and $\Gamma[\tilde{A},\tilde{K},\tilde{f}](y) \equiv 0$ are called strongly equivalent if there exist pointwise nonsingular matrix functions $E \in \mathbf{C}(\mathbb{I},\mathbb{R}^{r\times r})$ and $F \in \mathbf{C}(\mathbb{I},\mathbb{R}^{r\times r})$ such that

$$\begin{aligned}
\tilde{A}(t) &= E(t)A(t)F(t),\\
\tilde{K}(t,s,y(s)) &= E(t)K(t,s,F(s)y(s)), \qquad (4)\\
\tilde{f}(t) &= E(t)f(t)
\end{aligned}$$

If this is the case, we write $[A,K,f] \sim [\tilde{A},\tilde{K},\tilde{f}]$. Moreover, if $E \in \mathbf{C}^{\nu}(\mathbb{I},\mathbb{R}^{r\times r})$ and $F \in \mathbf{C}^{\nu}(\mathbb{I},\mathbb{R}^{r\times r})$ we can write "strongly ν-equivalent" instead of "strongly equivalent".

Theorem 3.1 Let $[A,K,f] \sim [\tilde{A},\tilde{K},\tilde{f}]$. Then, x is a solution of $\Gamma[A,K,f](x) \equiv 0$ iff $F^{-1}x$ is a solution of $\Gamma[\tilde{A},\tilde{K},\tilde{f}](y) \equiv 0$. This means if one of the strongly equivalent systems has a unique solution, another, has also a unique solution.

Proof. Let $\Gamma[A,K,f](x) \equiv 0$. Multiplying this equation by E and using $x \equiv FF^{-1}x$, we obtain $\Gamma[EA,EK,Ef](FF^{-1}x) \equiv 0$. Hence, for $t \in \mathbb{I} := [0,T]$,

$$E(t)A(t)F(t)\underbrace{F^{-1}(t)x(t)} + \int_0^t E(t)K(t,s,F(s)\underbrace{F^{-1}(s)x(s)})ds - E(t)f(t) = 0,$$

and thus $\Gamma[\tilde{A},\tilde{K},\tilde{f}](F^{-1}x) \equiv 0$, (It should not be confused with the substitution rule in calculus). Conversely, suppose $\Gamma[\tilde{A},\tilde{K},\tilde{f}](F^{-1}x) \equiv 0$. We can multiply this equation by E^{-1} to obtain $\Gamma[A,K,f](x) \equiv 0$, which proves the theorem.

This theorem is not true for strongly equivalent time variant DAEs, therefore, Kunkel and Mehrmann [8] has defined globally equivalent concept.

4. Discretised Collocation Methods on Piecewise Polynomial Space

Let $\mathbb{I}_h := \{t_n : 0 = t_0 < t_1 < ... < t_N = T\}$, be a given (not necessarily uniform) partition of $\mathbb{I}$, and set $\sigma_n := (t_n, t_{n+1}]$, $\bar{\sigma}_n := [t_n, t_{n+1}]$, with $h_n = t_{n+1} - t_n$ for $n = 0,...,N-1$ and diameter of this partition be $h = max\{h_n : 0 \le n \le N\}$.

Definition 4.1 [1] For a given mesh $\mathbb{I}_h$, the piecewise polynomial space, with $\mu > 0$, $-1 \le d \le \mu$, is given by

$$\mathbb{S}_{\mu}^{(d)}(\mathbb{I}_h) := \{v \in C^d(\mathbb{I}) : v|_{\bar{\sigma}_n} \in \pi_{\mu} (n = 0,1,...,N-1)\}. \qquad (5)$$

Here, π_{μ} denotes the space of (real) polynomials of degree not exceeding μ (also, $C^{-1}(\mathbb{I})$ is the space of absolutely continuous functions).

In this paper, we only consider $d = -1$ which the corresponding spaces is called discontinuous space. By defining $u_n = u_h|_{\bar{\sigma}_n} \in (\pi_{m-1})^r$, the dense output of approximate solution $u_h \in \mathbb{S}_{m-1}^{(-1)}(\mathbb{I}_h)$ can be obtained by

$$u_n(t_n + sh_n) = \sum_{j=1}^{m} L_j(s)U_{n,j}, \quad s \in (0,1], \qquad (6)$$

where the polynomials

$$L_j(v) := \prod_{\substack{k=1\\k\neq j}}^{m} \frac{v - c_k}{c_j - c_k}, \quad j = 1,\dots,m,$$

denote the Lagrange fundamental polynomials with respect to the distinct collocation parameters $0 < c_1 < c_2 < \dots < c_m \le 1$. The unknowns $U_{n,i} := u(t_{n,i})$, can be obtained by applying discretised discontinuous collocation methods (DDCM). Implementing DDCM to the IAE (2), we obtain $U_{n,i}$ by solving following system (see [1, 12]):

$$A(t_{n,i})U_{n,i} + F_{n,i} + h\sum_{j=1}^{m} a_{ij}K(t_{n,i}, t_{n,j}, U_{n,j}) = f(t_{n,i}) \qquad (7)$$

for $i = 1,\dots,m$, where the lag term is defined by

$$F_{n,i} = h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j K(t_{n,i}, t_l + sh_l, \sum_{j=1}^{m} L_j(s)U_{l,j}),$$

and $t_{n,i} = t_n + c_i h$. Here, $b_j = \int_0^1 L_j(s)ds$ and

$$a_{ij} = \int_0^{c_i} L_j(s)ds$$

for $i = 1,\dots,m$ and $j = 1,\dots,m$, [1,12].

Theorem 4.1. Let u_h be the unique approximate solution of applying DDCM to the IAE $[A,K,f] \sim [\tilde{A},\tilde{K},\tilde{f}]$. Then, $\tilde{u}_h$, the approximate solution of applying DDCM to the IAE $\Gamma[\tilde{A},\tilde{K},\tilde{f}](y) \equiv 0$, is unique and

$$\tilde{u}_h(t_{n,i}) = F^{-1}(t_{n,i})u_h(t_{n,i}), \quad n = 0,\dots,N-1, \qquad (8)$$

where $i = 1,\dots,m$.

Proof. The proof is by induction on n. Suppose $n = 0$. We show the system

$$A(t_{0,i})\tilde{u}_h(t_{0,i}) + h\sum_{j=1}^{m} a_{ij}\tilde{K}(t_{0,i}, t_{0,j}, \tilde{u}_h(t_{0,j})) = \tilde{f}(t_{0,i}), \qquad (9)$$

for $i = 1,\dots m$, has a unique solution. Multiplying the left-hand side of equation (9) by $E^{-1}(t_{0,i})$, and left-hand side of the term $\tilde{u}_h(t_{0,j})$ by $F^{-1}(t_{0,i})F(t_{0,i})$, we obtain

$$\overbrace{E^{-1}(t_{0,i})\tilde{A}(t_{0,i})F^{-1}(t_{n,i})}\underbrace{F(t_{0,i})\tilde{u}_h(t_{0,i})}$$

$$h\sum_{j=1}^{m} a_{ij}\overbrace{E^{-1}(t_{0,i})\tilde{K}(t_{0,i}, t_{0,j}, F^{-1}(t_{0,j})\underbrace{F(t_{0,j})\tilde{u}_h(t_{0,j})})} \qquad (10)$$

$$\overbrace{E^{-1}(t_{0,i})\tilde{f}(t_{0,i})}, \qquad i = 1,\dots m.$$

Setting $X_i := F(t_{0,i})\tilde{u}_h(t_{0,i})$, the system (10) can be written as

$$A(t_{0,i})X_i + h\sum_{j=1}^{m} a_{ij}K(t_{0,i}, t_{0,j}, X_j) = f(t_{0,i}), \quad i = 1,\dots m. \tag{11}$$

which has unique solution $X_i = u_h(t_{0,i}) = F(t_{0,i})\tilde{u}_h(t_{0,i})$, by hypotheses of the theorem. Now, assume that (8) is true for n. we will show that it is true for $n+1$. Hence we show that $\tilde{u}_h(t_{n+1,i}) = F^{-1}(t_{n+1,i})u_h(t_{n+1,i})$ is a unique solution of the system

$$\begin{aligned} &\tilde{A}(t_{n+1,i})\tilde{u}_h(t_{n+1,i}) + h\sum_{j=1}^{m} a_{ij}\tilde{K}(t_{n+1,i}, t_{n+1,j}, \tilde{u}_h(t_{n+1,j})) \\ &= -h\sum_{l=0}^{n}\sum_{j=1}^{m} b_j \tilde{K}(t_{n+1,i}, t_l + c_j h_l, \tilde{u}_h(t_{l,j})) + \tilde{f}(t_{n+1,i}), \end{aligned} \tag{12}$$

for $i = 1,\dots m.$ Multiplying the left-hand side of equation (12) by $E(t_{n+1,i})$, and left-hand sides of the terms $\tilde{u}_h(t_{l,j})$ by $F^{-1}(t_{l,i})F(t_{l,i})$, for $l = 0,\dots,n+1$, we obtain

$$\begin{aligned} &\overbrace{E(t_{n+1,i})\tilde{A}(t_{n+1,i})F^{-1}(t_{n+1,i})}\,\underbrace{F(t_{n+1,i})\tilde{u}_h(t_{n+1,i})} \\ &h\sum_{j=1}^{m} a_{ij}\,\overbrace{E(t_{n+1,i})\tilde{K}(t_{n+1,i}, t_{n+1,j}, F^{-1}(t_{n+1,j})\,\underbrace{F(t_{n+1,j})\tilde{u}_h(t_{n+1,j})})} \\ &h\sum_{l=0}^{n}\sum_{j=1}^{m} b_j\,\overbrace{E(t_{n+1,i})\tilde{K}(t_{n+1,i}, t_l + c_j h_l, F^{-1}(t_{l,j})\,\underbrace{F(t_{l,j})\tilde{u}_h(t_{l,j})})} \\ &+\overbrace{E(t_{n+1,i})\tilde{f}(t_{n+1,i})}, \qquad i = 1,\dots m. \end{aligned} \tag{13}$$

Setting $X_i := F(t_{n+1,i})\tilde{u}_h(t_{n+1,i})$, and using the induction hypothesis $\tilde{u}_h(t_{l,i}) = F^{-1}(t_{l,i})u_h(t_{l,i})$ for $l = 0,\dots,n$ the system (13) can be written as

$$\begin{aligned} &A(t_{n+1,i})X_i + h\sum_{j=1}^{m} a_{ij}K(t_{n+1,i}, t_{n+1,j}, X_j) \\ &= -h\sum_{l=0}^{n}\sum_{j=1}^{m} b_j K(t_{n+1,i}, t_l + c_j h_l, u_h(t_{l,j})) + f(t_{n+1,i}), \end{aligned} \tag{14}$$

which has unique solution $X_i = u_h(t_{n+1,i}) = F(t_{n+1,i})\tilde{u}_h(t_{n+1,i})$, by hypotheses of the theorem. This completes the proof of the theorem.

Corollary 4.1 Let u_h and $\tilde{u}_h$ be the approximate solutions of applying DDCM to the linear IAEs $\Gamma[A,K,f](y) \equiv 0$ and $\Gamma[\tilde{A},\tilde{K},\tilde{f}](\tilde{y}) \equiv 0$, which have unique solution y and $\tilde{y}$, respectively. Let $[A,K,f] \sim [\tilde{A},\tilde{K},\tilde{f}]$. Then, there exist two positive constants c_1 and c_2 such that

$$c_1 \|\tilde{y} - \tilde{u}\| \le \|y - u\| \le c_2 \|\tilde{y} - \tilde{u}\|$$

where $\| f \|$ is the max norm, $\max_{i,n}\{| f(t_{n,i})|\}$, for $n = 0,\dots,N-1$, $i = 1,\dots,m.$

Proof. By Theorems 3.1 and 4.1, there exists a pointwise nonsingular matrix function $F \in \mathbf{C}(\mathbb{I}, \mathbb{R}^{r\times r})$ such that $\tilde{y}(t) = F^{-1}(t)y(t)$ and $\tilde{u}_h(t_{n,i}) \le F^{-1}(t_{n,i})u_h(t_{n,i})$. Therefore,

$$\|\tilde{y} - \tilde{u}\| \le \|F^{-1}\| \|y - u\|$$

and

$$\|y - u\| \le \|F\| \|y - u\|.$$

Since, $\det F(t) \neq 0, \forall t \in \mathbb{I}$ and $F^{-1}(t) = \dfrac{adjF(t)}{\det F(t)}$, hence $F^{-1} \in \mathbf{C}(\mathbb{I}, \mathbb{R}^{r\times r})$. Thus, both functions $F(t)$ and $F^{-1}(t)$ are bounded and there exist real numbers $c_1 > 0$ and $c_2 > 0$ such that $\| F \| < c_2$ and $\| F^{-1} \| < \dfrac{1}{c_1}$. Consequently, we have

$$c_1 \|\tilde{y} - \tilde{u}\| \le \|y - u\|$$

and

$$\| y - u \| \le c_2 \| \tilde{y} - \tilde{u} \|,$$

which prove the theorem.

Corollary 4.1 shows that the approximate solutions of the stated methods for strongly equivalent systems $\Gamma[A,K,f](y) \equiv 0$ and $\Gamma[\tilde{A},\tilde{K},\tilde{f}](\tilde{y}) \equiv 0$, are of the same order, which is the key point in the numerical analysis of IAEs.

5. IAEs of Index 1

By using Theorem 3.1, the linear IAEs (2) of index 1, can be divided to two categories.

Theorem 5.1. For all linear IAEs of index 1:

(I) there exists a pointwise nonsingular matrix function $\hat{k}$ of dimension $r \times r$, and a vector function $\hat{f}$, such that

$$[A,k,f] \sim \left[O, \hat{k}, \hat{f}\right] \tag{15}$$

where O is a zero matrix function, or

(II)there exist matrix functions $\tilde{k}_{ij}$ and $\tilde{f}_j$, for $i, j \in \{1,2\}$, such that

$$[A,k,f]\sim \left[\begin{pmatrix}\mathbf{I} & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}\tilde{k}_{11}(t,s) & \tilde{k}_{12}(t,s)\\ \tilde{k}_{21}(t,s) & \tilde{k}_{22}(t,s)\end{pmatrix},\begin{pmatrix}\tilde{f}_1(t)\\ \tilde{f}_2(t)\end{pmatrix}\right] \tag{16}$$

where $\tilde{k}_{22}(t,t)$ is a pointwise nonsingular matrix function of dimension $(r-r_1)\times(r-r_1)$.

Proof. If, $A\equiv 0.$ then, $\det k(t,t)\neq 0$ on $\mathbb{I}$, and this is the first kind Volterra integral equation, (case (I)). Thus, suppose $0<rankA=r_1=const<r.$ Therefore, there exist nonsingular matrix functions E_0 and F_0 such that

$$\tilde{A}=E_0(t)A(t)F_0(t)=\begin{pmatrix}\mathbf{I} & 0\\ 0 & 0\end{pmatrix}, \tag{17}$$

since $rankA(t)=const.$ Hence,

$$\tilde{A}^{-}=\begin{pmatrix}\mathbf{I} & 0\\ 0 & 0\end{pmatrix},\quad A^{-}=F_0\tilde{A}^{-}E_0.$$

Let

$$\tilde{k}(t,s)=E_0(t)k(t,s)F_0(s)=\begin{pmatrix}\tilde{k}_{11}(t,s) & \tilde{k}_{12}(t,s)\\ \tilde{k}_{21}(t,s) & \tilde{k}_{22}(t,s)\end{pmatrix},$$

and

$$\tilde{f}(t)=E_0(t)f(t)=\begin{pmatrix}\tilde{f}_1(t)\\ \tilde{f}_2(t)\end{pmatrix}$$

Then, we have

$$E_0(t)(A(t)+(\mathbf{I}-AA^{-})k(t,t))F_0(t)=\tilde{A}+(\mathbf{I}-\tilde{A}\tilde{A}^{-})\tilde{k}(t,t)$$
$$=\begin{pmatrix}\mathbf{I} & 0\\ \tilde{k}_{11}(t,t)\tilde{k}_{21}(t,t) & \tilde{k}_{22}(t,t)\end{pmatrix}.$$

On the other hand, we see that using the definition 2.2, the matrix function $A(t)+(\mathbf{I}-AA^{-})k(t,t)$ is nonsingular on $\mathbb{I}$. Hence $\tilde{k}_{22}(t,t)$ is also a nonsingular matrix function on $\mathbb{I}$.

Remark 5.1 It is straightforward to see that the strongly equivalency of the equation (16) can be replaced by the strongly ν-equivalency if the ν th derivatives of the matrix functions A, k and f with respect to their variables, are continuous.

6. Global Convergence Analysis

The global convergence of the DDCM for the cases (I) and (II) has been studied in [1] (Section 2.4) as system of dimension $r=1$, and in [7] as system of dimension $r=2$. For system of arbitrary dimension, one can see [12] (Theorem 2, with $m_1=\infty$). The complete global convergence analysis of the method DDCM is investigated [10]. Another proof can be obtained as follow:

Theorem 4.1. Let the linear system (2) be of index 1 and

$$f(t)\in\mathbf{C}^{m+1}(\mathbb{I},\mathbb{R}^r),\quad A(t)\in\mathbf{C}^{m}(\mathbb{I},\mathbb{R}^{r\times r}),$$
$$k(t,s),\frac{\partial k(t,s)}{\partial t}\in\mathbf{C}^{m}(\mathbb{D},\mathbb{R}^{r\times r}),$$

for $m\in\mathbb{N}$. Then the approximate solution of applying the DDCM for sufficiently small h, say, u_h with distinct collocation parameters $c_1,\ldots,c_m\in(0,1]$ converges to the solution y, for $m\geq 1$, as $h\to 0$, with $Nh<const.$, if and only if

$$-1\leq\lambda:=(-1)^m\prod_{i=1}^{m}\frac{1-c_i}{c_i}\leq 1.$$

Moreover, the following error estimates holds:

$$\|y-u_h\|_\infty=\begin{cases}\mathcal{O}(h^m), & if\ \lambda\in[-1,1),\\ \mathcal{O}(h^{m-1}), & if\ \lambda=1,\end{cases}$$

as $h\to 0$, with $Nh<const.$

Proof. Using Corollary 4.1, and Theorem 5.1 it is sufficient to prove this theorem for the cases (I) and (II) of the Theorem 5.1. The case (I) can be obtained by taking $m_1=\infty$ in [12] (Theorem 2), for dimension $r=1$, see [1] (Section 2.4). For case (II), the required analysis exists only for dimension $r=2$, [7], and a similar proof can be provided for arbitrary dimension.

Remark 6.1 The supperconvergence result of [7] can be expressed for the case (II), as a direct result of corollary 4.1 (see also [10]).

7. Nonlinear Systems

Assume that the system (2) has a unique solution $y\in\mathbf{C}(\mathbb{I},\mathbb{R}^r)$. This assumption is important, since many nonlinear integral equations are ill-posed. Suppose K has continuous derivative with respect to s. To generalize the index definitions for nonlinear systems given in [12, 13], we introduce following definition

Definition 7.1 We say that the index for the nonlinear system (2) is ν, if there exists a neighborhood of the exact solution y, $N_\epsilon(y)=\{\eta\in\mathbf{C}^\nu(\mathbb{I},\mathbb{R}^r):\|\eta-y\|\leq\epsilon\}$, $\epsilon>0$, in which the index of linear system

$$A(t)y(t)+\int_0^t K_y(t,s,\eta(s))u(s)ds=R(t) \tag{18}$$

be ν, for all $\eta\in N_\epsilon(y)$ and for a function $R\in\mathbf{C}^\nu(\mathbb{I},\mathbb{R}^r)$. Moreover, we say that the index of (A,K) is ν, if there exists a neighborhood of the exact solution y, $N_\epsilon(y)$ in which the index of (A,K_u) be ν, for all $\eta\in N_\epsilon(y)$.

By using the definition (2.2), if the index of the system (18) for one $R\in\mathbf{C}^\nu(\mathbb{I},\mathbb{R}^r)$ be ν for other $R\in\mathbf{C}^\nu(\mathbb{I},\mathbb{R}^r)$ is also ν. Thus, the definition of index is well defined. Now, we can analyze the systems of nonlinear IAEs using DDCM. We use Peano interpolation formula

$$k(t,s,y(s))y(t_n+sh)=\sum_{j=1}^{m}L_j(s)k(t,t_{n,j},y(t_{n,j}))+h^m S_{m,n}(t), \tag{19}$$

where $S_{m,n}(t)$ is the Peano reminder term, to obtain

$$\begin{aligned} &A(t_{n,i})y(t_{n,i})\\ &+h\sum_{l=0}^{n-1}\left(\sum_{j=1}^{m}b_jK(t_{n,i},t_{l,j},y(t_{l,j}))+\int_0^1 h^{m+1}S_{m,n}(t_{n,i})ds\right)\\ &+h\sum_{j=1}^{m}a_{i,j}K(t,t_{n,j},y(t_{n,j}))+h\int_0^{c_i}h^{m+1}S_{m,n}(t_{n,i})=f(t_{n,i}) \end{aligned} \tag{20}$$

from (2). Using mean value theorem, there exists $\theta(s)$ between $y(s)$ and $u(s)$ such that

$$K(t,s,u_h(s))-K(t,s,y(s))=K_u(t,s,\theta(s))(u_h(s)-y(s)).$$

Subtracting system (20) from (7) we obtain

$$\begin{aligned} A(t_{n,i})e(t_{n,i})+h\sum_{j=0}^{m}a_{i,j}K_u(t,t_{n,j},\eta(t_{n,j}))=\mathcal{O}(h^{m+1})\\ +h\sum_{l=0}^{n-1}\sum_{j=0}^{m}b_jK_u(t_{n,i},t_{l,j},\theta(t_{l,j}))e(t_{l,j}) \end{aligned} \tag{21}$$

where, $e=u_h-y$. Note that, if we apply the DDCM to the linear system

$$A(t)y(t)+\int_0^t K_y(t,s,\theta(s))u(s)ds=R(t), \tag{22}$$

then we will obtain a linear system of the error function similar to the system (21). Thus, the order of the error functions in the nonlinear systems of index ν is equal to its corresponding linear system of index ν.

Remark 7.1 Note that, in the above argument, we do not know anything about the continuity or the differentiability of the function θ, hence of the function $K(t,s)=K_y(t,s,\theta(s))$ with respect to s. However, it does not damage our expressed reasons, since in the above analysis we only need the $m+1$ times differentiability of the solution (to use the Peano interpolation formula), which follows from the assumption $K(t,s,y)\in\mathbf{C}^{m+1}(\mathbb{D}\times\mathbb{R}^r,\mathbb{R}^r)$.

Now, we can state the following theorem which is the direct result of the expressed facts.

Theorem 7.1. Let the nonlinear system (2) be index 1 and

$$f(t)\in\mathbf{C}^{m+1}(\mathbb{I},\mathbb{R}^r),\quad A(t)\in\mathbf{C}^{m}(\mathbb{I},\mathbb{R}^{r\times r}),$$
$$K(t,s,y(s)),\in\mathbf{C}^{m+1}(\mathbb{D}\times\mathbb{R}^r,\mathbb{R}^r),$$

for $m\in\mathbb{N}$. Then the approximate solution of applying the DDCM for sufficiently small h, say, u_h with distinct collocation parameters $c_1,\dots,c_m\in(0,1]$ converges to the solution y, for $m\geq 1$, as $h\to 0$, with $Nh<const.$, if and only if

$$-1\leq\lambda:=(-1)^m\prod_{i=1}^{m}\frac{1-c_i}{c_i}\leq 1.$$

Moreover, the following error estimates holds:

$$\|y-u_h\|_\infty=\begin{cases}\mathcal{O}(h^m), & if\ \lambda\in[-1,1),\\ \mathcal{O}(h^{m-1}), & if\ \lambda=1,\end{cases}$$

as $h\to 0$, with $Nh<const.$

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