



Pseudo similar intuitionistic fuzzy matrices

T. Gandhimathi¹, A. R. Meenakshi²

¹Department of Mathematics, P.A. College of Engineering and Technology, Pollachi-642 002, India

²Department of Mathematics, Karpagam University, Coimbatore-642 021, India

Email address:

gandhimahes@rediffmail.com (T. Gandhimathi)

To cite this article:

T. Gandhimathi, A. R. Meenakshi. Pseudo Similar Intuitionistic Fuzzy Matrices. *Applied and Computational Mathematics*. Special Issue: New Advances in Fuzzy Mathematics: Theory, Algorithms, and Applications. Vol. 4, No. 1-2, 2015, pp. 15-19.
doi: 10.11648/j.acm.s.2015040102.14

Abstract: In this paper, we shall define Pseudo Similarity and Semi Similarity for Intuitionistic Fuzzy Matrix (IFM) and prove that the Pseudo similarity relation on a pair of IFMs is inherited by all its powers and their transposes are similar. Also we exhibit that the Pseudo similarity relation preserve regularity and impotency of their matrices.

Keywords: Fuzzy matrix, Intuitionistic Fuzzy Matrix, Pseudo Similar, Semi Similar

1. Introduction

We deal with fuzzy matrices that is, matrices over the fuzzy algebra FM and FN with support $[0,1]$ and fuzzy operations $\{+, \cdot\}$ defined as $a+b=\max\{a,b\}$, $a.b = \min\{a,b\}$ for all $a,b \in FM$ and $a + b = \min\{a, b\}$, $a \cdot b = \max\{a, b\}$ for all $a, b \in FN$. A matrix $A \in F^{M \times N}$ is said to be regular if there exists $X \in F^{N \times M}$ such that $AXA = A$, X is called a generalized inverse (g-inverse) of A . In [4], Kim and Roush have developed the theory of fuzzy matrices, under max min composition analogous to that of Boolean matrices. Cho [2] has discussed the consistency of fuzzy matrix equations. For more details on fuzzy matrices one may refer [6]. Atanassov has introduced and developed the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [1]. Basic properties of intuitionistic fuzzy matrices as a generalization of the results on fuzzy matrices have been derived by Pal and Khan [5]. In our earlier work, we have discussed on regularity and idempotency of IFMs[8]. The Pseudo Similarity in semi groups of fuzzy matrices were discussed by Chen and Meenakshi[3,7]

In this paper, pseudo similarity and semi similarity of an intuitionistic fuzzy matrix are discussed.

2. Preliminaries

In this paper, we are concerned with fuzzy matrices, that is matrices over a fuzzy algebra FM (FN) with support $[0,1]$, under maxmin (minmax) operations and the usual

ordering of real numbers. Let $(IF)^{M \times N}$ be the set of all intuitionistic fuzzy matrices of order $m \times n$, $F^{M \times N}$ be the set of all fuzzy matrices of order $m \times n$, under the maxmin composition and $F^{N \times M}$ be the set of all fuzzy matrices of order $n \times m$, under the minmax composition.

If $A = (a_{ij}) \in (IF)^{M \times N}$, then $A = \left(\langle a_{ij\mu}, a_{ij\nu} \rangle \right)$, where $a_{ij\mu}$ and $a_{ij\nu}$ are the membership values and non membership values of a_{ij} in A respectively with respect to the fuzzy sets μ and ν , maintaining the condition $0 \leq a_{ij\mu} + a_{ij\nu} \leq 1$.

We shall follow the matrix operations on intuitionistic fuzzy matrices as defined in [8].

For $A, B \in (IF)^{M \times N}$, then

$$A + B = \left(\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\} \rangle \right)$$

$$AB = \left(\left\langle \max_k \min\{a_{ik\mu}, b_{kj\mu}\}, \min_k \max\{a_{ik\nu}, b_{kj\nu}\} \right\rangle \right)$$

Let us define the order relation on $(IF)^{M \times N}$ as ,

$$A \leq B \Leftrightarrow a_{ij\mu} \leq b_{ij\mu} \text{ and } a_{ij\nu} \geq b_{ij\nu}, \text{ for all } i \text{ and } j.$$

Definition 2.1[5]

An $A \in (IF)^{M \times N}$ is said to be regular if there exists

$X \in (IF)_{n \times m}$ satisfying $AXA = A$ and X is called a generalized inverse (g-inverse) of A which is denoted by \bar{A} .

Let $A\{1\}$ be the set of all g-inverses of \bar{A} .

Definition 2.2[8]

A Matrix $A \in (IF)_n$ is said to be invertible if and only if there exists $X \in (IF)_n$ such that $AX = XA = I_n = \langle I_n^M, I_n^N \rangle$

where I_n is the identity matrix in $(IF)_n$

Definition 2.3[8]

A square intuitionistic fuzzy matrix is called intuitionistic fuzzy permutation matrix, if every row and column contains exactly one $\langle 1, 0 \rangle$ and all other entries are $\langle 0, 1 \rangle$. Let P_n be the set of all $n \times n$ permutation matrices in $(IF)_n$.

Definition 2.4[8]

Let $A \in (IF)_n$. Then

- (i) A is reflexive $\Leftrightarrow A \geq I_n$.
- (ii) A is symmetric $\Leftrightarrow A = A^T$.
- (iii) A is transitive $\Leftrightarrow A^2 \leq A$.
- (iv) A is idempotent $\Leftrightarrow A = A^2$

3. Pseudo Similar Intuitionistic Fuzzy Matrices

In this section, the pseudo similarity and semi similarity of an intuitionistic fuzzy matrix are discussed.

Definition 3.1

$A \in (IF)_m$ and $B \in (IF)_n$ are said to be Pseudo-Similar and denote it by $A \approx B$ if there exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A = XBY$, $B = YAX$ and $X = XYX$.

Definition 3.2

$A \in (IF)_m$ and $B \in (IF)_n$ are said to be Semi-Similar and denote it by $A \approx B$ if there exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A = XBY$ and $B = YAX$.

Definition 3.3

For $A \in (IF)_n$, the least $d > 0$ such that $A^{k+d} = A^k$ for some integer k , is called the period of A .

Theorem 3.1

Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent.

- (i) $A \approx B$
- (ii) There exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A = XBY$ and $B = YAX$ and $XY \in (IF)_m$ is idempotent.
- (iii) There exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A = XBY$ and $B = YAX$ and $YX \in (IF)_n$ is idempotent.

Proof:

(i) \Rightarrow (ii) and (i) \Rightarrow (iii) are trivial, since $X = XYX$ implies $XY \in (IF)_m$ and $YX \in (IF)_n$ are idempotent matrices.

(ii) \Rightarrow (i): $A = XBY = (XY)A(XY) = (XYX)B(YXY)$. Similarly, $B = YAX = (YXY)A(XYX)$

Put $X' = XYX$ and $Y' = YXY$. Then $A = X'BY'$ and $B = Y'AX'$.

Further using XY is idempotent, we get

$$X'Y' = (XYX)(YXY) = XY \text{ and}$$

$$(X'Y')(X'Y') = (XY)(XY) = X'Y'.$$

Thus $X'Y'$ is idempotent.

Set $X'' = X'Y'X'$ and $Y'' = Y'X'Y'$. Then

$$A = X'BY'$$

$$= X'(Y'AX')Y'$$

$$= X'Y'(X'BY')X'Y'$$

$$= (X'Y'X')B(Y'X'Y') = X''BY''$$

$$\text{Similarly, } B = Y''AX''.$$

By using $X'Y'$ is idempotent, we have

$$X''Y''X'' = (X'Y'X')(Y'X'Y')(X'Y'X')$$

$$= X'Y'X'$$

$$= X''.$$

Therefore $A \approx B$. Thus (i) holds.

(iii) \Rightarrow (i): This can be proved in the same manner and hence omitted.

Theorem 3.2:

Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent.

- (i) $A \approx B$
- (ii) There exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A = XBY$ and $B = YAX$ and $(XY)^k \in (IF)_m$ is idempotent for some odd $k \in \mathbb{N}$.
- (iii) There exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A = XBY$ and $B = YAX$ and $(YX)^k \in (IF)_n$ is idempotent for some odd $k \in \mathbb{N}$.

Proof:

(i) \Rightarrow (ii): Follows from Theorem (3.1).

(ii) \Rightarrow (i): Let $k = 2r+1$ with $r \in \mathbb{N}$, Since $A = XBY$ and $B = YAX$, we have

$$A = X(YAX)Y$$

$$= (XY)XBY(XY)$$

$$= (XY)X(YAX)Y(XY)$$

$$= (XY)^2A(XY)^2$$

$$= (XY)^2XBY(XY)^2$$

Thus proceeding we get,

$$A = (XY)^r XBY (XY)^r.$$

$$\text{Similarly, } B = Y(XY)^r A (XY)^r X.$$

$$\text{Set } X' = (XY)^r X \text{ and } Y' = Y(XY)^r. \text{ Then } A = X'BY',$$

$$B = Y'AX' \text{ and } X'Y' = (XY)^r XY (XY)^r$$

$$= (XY)^{2r+1}$$

$$= (XY)^k \text{ is idempotent.}$$

Hence by Theorem (3.1), $A \approx B$. Thus (i) holds.

(i) \Rightarrow (iii): Follows from Theorem (3.1)

(iii) \Rightarrow (i): Let $k = 2r+1$ with $r \in \mathbb{N}$, Since $A = XBY$ and $B = YAX$, we have

$$A = X(YX)^r B (YX)^r Y$$

$$B = (YX)^r YAX (YX)^r$$

$$\text{Set } X' = X(YX)^r \text{ and } Y' = (YX)^r Y. \text{ Then } A = X'BY',$$

$$B = Y'AX'$$

$$\text{and } Y'X' = (YX)^r (YX) (YX)^r$$

$$= (YX)^{2r+1}$$

$$= (YX)^k \in (IF)_n \text{ is idempotent.}$$

Hence by Theorem (3.1), $A \approx B$. Thus (i) holds.

Theorem 3.3:

Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent.

- (i) $A \approx B$
- (ii) There exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that

$A=XB Y$ and $B=YAX$ and $(XY) \in (IF)_m$ is periodic with even order.

- (iii) There exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A=XB Y$ and $B=YAX$ and $(YX) \in (IF)_n$ is periodic with even order.

Proof:

(i) \Rightarrow (ii): Follows from Theorem (3.1), since $XY \in (IF)_{m \times n}$ is idempotent, it follows that XY is periodic with even order.

(ii) \Rightarrow (i): Suppose $A=XB Y$ and $B=YAX$ and XY is periodic with even order $2k(k \in \mathbb{N})$ say, then $(XY)^{2k}=XY$.

Hence $(XY)^{2k-1}(XY)=XY$

$$\text{Further } (XY)^{2k-1}(XY)^{2k-1} = (XY)^{2k-1}XY(XY)^{2k-1} \\ = (XY)^{2k-1}.$$

Thus $(XY)^{2k-1}$ is idempotent.

Therefore $A \simeq B$, by Theorem (3.2). Thus (i) holds.

(i) \Leftrightarrow (iii): This can be proved in the same manner and hence omitted.

Theorem 3.4:

Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent.

- (i) $A \simeq B$
 (ii) There exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A=XB Y$, $B=YAX$, $X=XYX$ and $Y=YXY$
 (iii) There exist $X \in (IF)_{m \times n}$ and $Y, Z \in (IF)_{n \times m}$ such that $A=XB Y$, $B=ZAX$, $X=XYX = XZX$

Proof:

(i) \Rightarrow (iii): Since $A \simeq B$, By Definition(3.1), there exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A=XB Y$, $B=YAX$, $X=XYX$. Let $Y=Z$ then $B=ZAX$ and $X=XZX$ as required. Thus (iii) holds.

(iii) \Rightarrow (ii): Suppose there exist $X \in (IF)_{m \times n}$ and $Y, Z \in (IF)_{n \times m}$ such that $A=XB Y$, $B=ZAX$, $X=XYX = XZX$, then

$$A=XB Y=X(ZAX)Y=(XZX)B(YXY) \\ =XB(YXY) \text{ and} \\ B=ZAX=Z(XBY)X=(XZX)A(XYX) \\ =(XZX)AX.$$

Set $Y'=YXY$ and $Z'=XZX$. Then

$$X=XYX=XY(XYX)=XY'X \text{ and} \\ X=XZX=XZ(XZX)=XZ'X.$$

In addition, we have $A=XB Y'$ and $B=Z'AX$.

Set $Y''=Z'XY'$. Then $XY''X=XZ'(XY'X)=XZ'X=X$ and $Y''XY''=Z'(XY'X)Y'=Z'XY'=Y''$.

We check that $XB Y''=XBZ'XY'$

$$=XZ'AXZ'XY' \\ =XZ'AXY' \\ =XB Y'=A,$$

$$Y''AX=Z'XY'XBY'X=Z'XB Y'X$$

$$=Z'AX=B.$$

Thus there exist $X \in (IF)_{m \times n}$, $Y'' \in (IF)_{n \times m}$ such that $A=XB Y''$, $B=Y''AX$, $X=XY''X$ and $Y''=Y''XY''$ as asserted and (ii) holds.

(ii) \Rightarrow (i): This is trivial.

Remark 3.1:

We observe that Theorem (3.4) infers the symmetry of Pseudo similarity relation, that is, $A \simeq B \Leftrightarrow B \simeq A$.

Theorem 3.5:

Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent.

- (i) $A \simeq B$
 (ii) $A^T \simeq B^T$
 (iii) $A^k \simeq B^k$ for any integer $k \geq 1$
 (iv) $PAP^T \simeq QBQ^T$ for some permutation matrices $P \in (IF)_m$ and $Q \in (IF)_n$

Proof:

(i) \Leftrightarrow (ii): This is direct by taking transpose on both sides of $A=XB Y$, $B=YAX$ and using $(A^T)^T=A$ and $(AX)^T=X^TA^T$.

(i) \Leftrightarrow (iii): (iii) \Rightarrow (i) is trivial. (i) \Rightarrow (iii) can be proved by induction on k . Since $A \simeq B$, there exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A=XB Y$, $B=YAX$, $X=XYX$. By Theorem(3.4), further we have $Y=YXY$. Let us assume that $A^r=XB^rY$, $B^r=YA^rX$ for some $r \in \mathbb{N}$.

$$A=XB Y \Rightarrow AXY=XB YXY \\ =XB Y = A, \\ XB^{r+1}Y = XB^rBY \\ =XB^rBY \\ =XB^r(YAXY) \\ =XB^rYA = A^{r+1}$$

Similarly it can be checked $YAr+1X = Br+1$.

Thus by induction for all positive integer k , we get, $Ak=XBkY$, $Bk=YAkX$ for $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such

that $X=XYX$. Hence $Ak \simeq Bk$.

Thus (i) \Rightarrow (iii) hold.

(i) \Leftrightarrow (iv): Suppose $A \simeq B$, then there exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that

$$A=XB Y, B=YAX, X=XYX.$$

$$A=XB Y \Rightarrow PAP^T = PXBYP^T$$

$$= (PXQ^T)(QBQ^T)(QYP^T)$$

$$B=YAX \Rightarrow QBQ^T = QYAXQ^T$$

$$= (QYP^T)(PAP^T)(PXQ^T).$$

Set $X'=PXQ^T$, $Y'=QYP^T$ then $X'Y'X'=X'$, $PAP^T=X'(QBQ^T)Y'$ and $QBQ^T=Y'(PAP^T)X'$.

Thus $PAP^T \simeq QBQ^T$. Conversely, if

$$PAP^T \simeq QBQ^T \text{ then as above,}$$

$$P^T(PAP^T)P \simeq Q^T(QBQ^T)Q \Rightarrow A \simeq B$$

Thus (i) \Leftrightarrow (iv) holds.

Theorem 3.6:

Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent.

- (i) $A \approx B$
- (ii) $A^T \approx B^T$
- (iii) $A^k \approx B^k$ for any integer $k \geq 1$
- (iv) $PAP^T \approx QBQ^T$ for some permutation matrices $P \in (IF)_m$ and $Q \in (IF)_n$

Proof:

This can be proved along the same manner as that of Theorem (3.5) and hence omitted.

Remark 3.2:

It is clear that Similarity \Rightarrow Pseudo similarity \Rightarrow Semi similarity but converse is not true.

Example 3.1:

Consider the IFMs,

$$A = \begin{bmatrix} \langle 0.5, 0 \rangle & \langle 0.5, 0 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle \end{bmatrix}$$

and

$$B = \begin{bmatrix} \langle 0.5, 0 \rangle & \langle 0, 1 \rangle \\ \langle 0.5, 0 \rangle & \langle 0, 1 \rangle \end{bmatrix}$$

with respect to

$$X = \begin{bmatrix} \langle 1, 0 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \langle 1, 0 \rangle & \langle 1, 0 \rangle \\ \langle 1, 0 \rangle & \langle 0, 0.5 \rangle \end{bmatrix} \in X\{1\}.$$

Here $A = XBY$ and $B = YAX$.

This implies that $A \approx B$ and $A \approx B$.

Example 3.2

Let us consider the IFMs

$$X = \begin{bmatrix} \langle 0.3, 0.3 \rangle & \langle 1, 0 \rangle \\ \langle 0.5, 0.5 \rangle & \langle 0.2, 0.2 \rangle \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \langle 1, 0 \rangle & \langle 1, 0 \rangle \\ \langle 0, 0.5 \rangle & \langle 0, 0.5 \rangle \end{bmatrix}$$

then $XYX \neq X$ and $YXY \neq Y$.

$$\text{For } A = \begin{bmatrix} \langle 0.5, 0.3 \rangle & \langle 0.2, 0.3 \rangle \\ \langle 0, 0.5 \rangle & \langle 0, 0.5 \rangle \end{bmatrix}$$

and

$$B = \begin{bmatrix} \langle 0.3, 0.3 \rangle & \langle 0.3, 0 \rangle \\ \langle 0.5, 0.5 \rangle & \langle 0.5, 0.5 \rangle \end{bmatrix},$$

$B = XAY$ and $A = YAX$.

Hence $A \approx B$ but $A \not\approx B$.

Next we can see that Pseudo Similarity relation preserve regularity and idempotency of matrices concerned.

Theorem 3.7:

Let $A \in (IF)_m$ and $B \in (IF)_n$ such that $A \approx B$. Then A is regular matrix $\Leftrightarrow B$ is regular matrix.

Proof:

Since $A \approx B$, by Theorem(3.4), there exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $A = XBY$, $B = YAX$, $X = XYX$ and $Y = YXY$. Suppose A is regular, then there exists $G \in (IF)_m$ such that $AGA = A$. Define $U = YGX$. Clearly $U \in (IF)_n$. Since,

$$AXY = XBYXY = XBY = A$$

$$= (XYX)BY$$

$$= XYA,$$

$$BUB = (YAX)(YGX)(YAX)$$

$$= Y(AXY)G(XYA)X$$

$$= YAGAX$$

$$= YAX = B.$$

Hence B is regular. Converse can be proved in the same manner.

Theorem 3.8:

Let $A \in (IF)_m$ and $B \in (IF)_n$ such that $A \approx B$. Then A is idempotent $\Leftrightarrow B$ is idempotent.

Proof:

Since $A \approx B$, by Theorem(3.4), there exist $X \in (IF)_{m \times n}$ and $Y \in (IF)_{n \times m}$ such that $B = YAX$ and $A = XBY \Rightarrow AXY = A = AYYX$.

Suppose A is idempotent, then $A^2 = A$, hence

$$B^2 = (YAX)(YAX) = Y(AXY)AX$$

$$= YAAAX$$

$$= YA^2X = YAX = B \text{ and } B \text{ is idempotent.}$$

Converse can be proved in the same manner.

4. Conclusion

In this paper, the Pseudo similarity and Semi similarity of IFMs are defined. The Pseudo similarity relation on a pair of IFMs is inherited by all its powers and the Pseudo similarity relation preserve regularity and impotency of matrices are discussed.

References

- [1] K. Atanassov, "Intuitionistic fuzzy sets", Fuzzy Sets and systems 20(1), (1986), 87-96.
- [2] H. H. Cho, "Regular fuzzy matrices and fuzzy equations", Fuzzy Sets and Systems, 105 (1999), 445 – 451.
- [3] Huanyin Chen, "Pseudo similarity in Semigroups", Semi group forum, 68(2004), 59-63
- [4] K. H. Kim and F. W. Roush, "Generalised fuzzy matrices", Fuzzy sets and Systems, 4 (1980), 293 – 315.
- [5] S. Khan and Anita Pal, "The Generalised inverse of intuitionistic fuzzy matrices", Journal of Physical science, Vol II (2007), 62-67.

- [6] A R. Meenakshi, "Fuzzy matrices: Theory and its applications", MJP Publishers, Chennai – 2008.
- [7] A R. Meenakshi, "Pseudo similarity in semigroups of fuzzy matrices", Proc. Int. Symp. on Semigroups and Appln., (2007), 64-73.
- [8] A R. Meenakshi and T. Gandhimathi, "On Regular intuitionistic Fuzzy matrices", Int.J. of Fuzzy Mathematics, Vol. 19, No. 2, (2011), 599-605.