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Some Triple Generations of the Lyons Sporadic Simple Group Ly

Malebogo Motalane[1]**, Zwelethemba Mpono**[2,*]

[1]Department of Mathematics and Applied Mathematics, Faculty of Science, University of Limpopo, Sovenga, South Africa
[2]Department of Mathematical Sciences, College of Science, Engineering and Technology, University of South Africa, Pretoria, South Africa

Email address:
john.motalane@ul.ac.za (Malebogo Motalane), mponoze@unisa.ac.za (Zwelethemba Mpono), mponoze@yahoo.com (Zwelethemba Mpono)
[*]Corresponding author

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Abstract: According to the classification theorem, the Lyons group Ly is one of the 26 sporadic simple groups and has order $51765179004000000 = 2^8{\cdot}3^7{\cdot}5^6{\cdot}7{\cdot}11{\cdot}31{\cdot}37{\cdot}67$. Since the completion of the classification of all finite simple groups, attention has now turned to other aspects e.g. generations of finite groups which entails determining elements which generate that finite group. As a finite nonabelian simple group, Ly can be generated by a minimum of two of its elements. We thus endeavour in the current study to determine some of the pairs of its elements of distinct prime orders from disctinct conjugacy classes with their product in another conjugacy class of elements of prime order which generate Ly and we call such generations triple generations. Triple generations of any finite group are used in the study of its symmetric genus, where the symmetric genus of a Hurwitz group G, of which Ly is known to be a Hurwitz group, is given by $1 + \frac{|G|}{84}$. If G is a finite group and lX, mY, nZ are conjugacy classes of elements of G, then G is said to be (l, m, n)-generated if $G = \langle x, y\rangle$ with $o(x) = l, o(y) = m$ and $o(xy) = n$. The number of distinct ordered pairs (x, y) satisfying $x \in lX, y \in mY$ such that $xy = z$, where $z \in nZ$ is an arbitrary class representative, is denoted by $\zeta_G(lX, mY, nZ)$ and is known as the *structure constant* of the group algebra $\mathbb{C}G$. The structure constants can be computed from the ordinary character table of G. We shall use the method of the structure constants to determine such generation and/or nongeneration. Thus the object in this paper is to study some of the triple generations of Ly which will thus pave the way towards the study of various combinations of three, four, five etc elements from distinct conjugacy classes which can generate Ly and lead to the ultimate determination of the maximum number of elements of Ly from distinct conjugacy classes of its elements which can generate Ly.

Keywords: (p, q, r)-Generations, Maximal Subgroups, Primes, Structure Constants, Conjugacy Class Fusions

1. Introduction

Richard Lyons [15] established the existence of a group of order $51765179004000000 = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$, which came to be known as the Lyons group, abbreviated as Ly. Its ordinary character table which was constructed by Lyons himself is found in [3] and its Brauer character tables corresponding to the various primes dividing its order are found in [9].

By [8], the information from the ordinary character table showed that the group (Ly) has no faithful matrix representation of degree less than 2480 in any field of characteristic 0, nor any permutation representation of degree less than eight million.

Every finite nonabelian simple group can be generated by a minimum of two of its elements. The Lyons group Ly is a finite nonabelian sporadic simple group with nine conjugacy classes of maximal subgroups which are listed in the Atlas [3] and its prime spectrum is given by $\{2, 3, 5, 7, 11, 31, 37, 67\}$.

Since the completion of the classification of all finite simple groups, attention has now turned to other aspects e.g. triple generations of finite groups which entails determining pairs of its elements which generate that finite group. We shall determine elements from distinct conjugacy classes of elements of Ly with product in another class, which generate Ly.

Thus in the current article, we shall study the triple generations of Ly through studying its (p, q, r)-generations for distinct primes $p, q, r \in \{2, 3, 5, 7, 11, 31, 37, 67\}$ and satisfying $p < q < r$.

In [5], Ganief and Moori studied the (p, q, r)-generations for the smallest Conway Group Co_3 and Moori in [10] studied the (p, q, r)-generations for the Janko groups J_1 and J_2. In [13, 14], Mpono studied the triple generations and the conjugacy class ranks of M_{24} respectively and Seretlo in [16] studied the group $5^3 \cdot L(3,5)$ which is a maximal subgroup of Ly.

For computations, we made use of the Computer Algebra System GAP [7] which is running on a LINUX machine in the Department of Mathematical Sciences at the University of South Africa, without which this work would have been almost impossible to do.

Throughout, G is a finite group and p, q, r denote distinct primes that divide the order of G unless specified otherwise. Reference to the various maximal subgroups of Ly should be understood to mean up to isomorphism.

All the relevant tables of the partial structure constants for both Ly and its contributing maximal subgroups together with the partial fusions into Ly of the relevant classes of its various maximal subgroups which will be used in each subsection, will be placed at the end of each relevant subsection.

This work forms part of the MSc dissertation [12] of the first author and was done under the supervision of the second author.

2. Preliminaries

If G is a finite group and lX, mY, nZ are conjugacy classes of elements of G, then G is said to be (l, m, n)-generated if $G = \langle x, y \rangle$ with $o(x) = l, o(y) = m$ and $o(xy) = n$. The number of distinct ordered pairs (x, y) satisfying $x \in lX, y \in mY$ such that $xy = z$, where $z \in nZ$ is an arbitrary class representative, is denoted by $\zeta_G(lX, mY, nZ)$ and is known as the *structure constant* of the group algebra $\mathbb{C}G$. The structure constants can be computed from the ordinary character table of G via the formula

$$\zeta_G(lX, mY, nZ) = \frac{|lX||mY|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1_G)}$$

and is independent of the choice of the representative $z \in nZ$. The number of such ordered pairs (x, y) which satisfy $G = \langle x, y \rangle$ will be denoted by $\zeta^*_G(lX, mY, nZ)$.

For H a subgroup of G containing $z \in nZ$, the number of distinct pairs (x, y) which satisfy $x \in lX, y \in mY$ and $\langle x, y \rangle \leq H$ is denoted by $\sum_H(lX, mY, nT)$, where nT is the conjugacy class of H containing $z \in nZ$. For $H_1, H_2, \ldots, H_m$ subgroups of G, the number of pairs (x, y) which generate a subgroup of some H_i for $1 \leq i \leq m$ will be denoted by $\sum(H_1 \cup H_2 \cup \cdots \cup H_m)$. We thus obtain that

$$\zeta^*_G(lX, mY, nZ)$$

satisfies

$$\zeta^*_G(lX, mY, nZ) = \zeta_G(lX, mY, nZ) - \sum(H_1 \cup H_2 \cup \cdots \cup H_m)$$

where $H_1, H_2, \ldots, H_m$ are the maximal subgroups of G containing $z \in nZ$. It thus follows that a group G admits an (l, m, n)-generation if and only if there are conjugacy classes of elements lX, mY, nZ of G for which $\zeta^*_G(lX, mY, nZ) > 0$. By [5], if G is an (l, m, n)-generated simple group, then we obtain that

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$$

Theorem 2.1. [1, 11] Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $((\underbrace{lX, lX, \ldots, lX}_{m-times}), (nZ)^m)$-generated.

Proof Suppose that $G = \langle x, y \rangle$ such that $x \in lX, y \in mY$ and $xy = z \in nZ$. Let

$$N = \langle x, x^y, x^{y^2}, \ldots, x^{y^{m-1}} \rangle$$

Thus N is a nontrivial normal subgroup of G so that $N = G$. Furthermore we obtain that $xx^yx^{y^2} \ldots x^{y^{m-1}} = x(yxy^{-1}y^2xy^{-2} \ldots (y^{m-2}xy^{2-m})(y^{m-1}xy^{1-m}) = (xy)^m = z^m$. Since $x^{y^i} \in lX$ for all $1 \leq i \leq m$, the result follows.

Corollary 2.1. [2, 4] Let G be a simple group such that G is $(2X, sY, tZ)$-generated. Then G is $(sY, sY, (tZ)^2)$-generated.

Proof This is [2, Lemma 2].

3. The $(2, q, r)$-Generations of Ly

For the $(2, q, r)$-generations of Ly, we shall consider all $q \in \{3, 5, 7, 11, 31, 37\}$ and all $r \in \{5, 7, 11, 31, 37, 67\}$.

3.1. The $(2, 3, r)$-Generations of Ly

For the $(2, 3, r)$-generations of Ly, we shall consider all $r \in \{5, 7, 11, 31, 37, 67\}$. The maximal subgroups $5^{1+4}_+{:}4S_6; 3^{2+4}{:}2{\cdot}A_5{\cdot}D_8$ and 37:18 have their relevant structure constants all zero and 67:22 is the only maximal subgroup which does not have elements of order 3. Thus they will not have any contributions in this case.

The maximal subgroups having any contributions are $G_2(5); 3{\cdot}McL{:}2; 5^3{\cdot}L_3(5); 2{\cdot}A_{11}$ and $3^5{:}(2 \times M_{11})$ because $G_2(5)$ contains elements of orders 5, 7 and 31, $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ contain elements of orders 5, 7 and 11, $5^3{\cdot}L_3(5)$ contains elements of orders 5 and 31, and $3^5{:}(2 \times M_{11})$ contains elements of orders 5 and 11. Table 1 gives the partial structure constants of Ly computed using GAP [7] that will be used and Tables 2 to 6 give the various partial structure constants for each contributing maximal subgroup.

Proposition 3.1. Ly is not $(2, 3, 5)$-generated.

Proof By Table 1, we have that $\zeta_{Ly}(2A, 3A, 5A) = 0 = \zeta_{Ly}(2A, 3A, 5B) = \zeta_{Ly}(2A, 3B, 5A)$, proving that Ly is not $(2A, 3A, 5A), (2A, 3A, 5B), (2A, 3B, 5A)$-generated. By the same Table 1, we obtain that $\zeta_{Ly}(2A, 3B, 5B) = 6875$. The maximal subgroup $5^3{\cdot}L_3(5)$ contains elements of order 5 but will not have any contributions here since all its relevant structure constants are zero. There are four contributing maximal subgroups viz. $G_2(5); 3{\cdot}McL{:}2; 2{\cdot}A_{11}$ and $3^5{:}(2 \times M_{11})$ having elements of order 5. By [4], [18], the class $5c$ of $G_2(5)$ is contained in one conjugate of $G_2(5)$ while $5d, 5e$ of $G_2(5)$ are each contained in three conjugates of $G_2(5)$. Again by [4], [18], the class $5b$ of $3{\cdot}McL{:}2$ is contained in 25 conjugates of $3{\cdot}McL{:}2$, $5b$ of $2{\cdot}A_{11}$ is contained in 75 conjugates of $2{\cdot}A_{11}$ while $5a$ of $3^5{:}(2 \times M_{11})$ is contained in 125 conjugates of $3^5{:}(2{\times}M_{11})$. We obtain from Table 7 that $\zeta_{G_2(5)}(2a, 3b, 5c) = 625 = \zeta_{G_2(5)}(2a, 3b, 5d) = \zeta_{G_2(5)}(2a, 3b, 5e)$ and by Table 8, we obtain that $\zeta_{3\cdot McL:2}(2a, 3d, 5b) = 125$. By Tables 10 and 12, we obtain that $\zeta_{2\cdot A_{11}}(2b, 3c, 5b) = 100, \zeta_{3^5:(2\times M_{11})}(2a, 3c, 5a) = 25$ and $\zeta_{3^5:(2\times M_{11})}(2a, 3d, 5a) = 20$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 5B) = 6875-625-3(625)-3(625)-25(125)-75(100) - 125(25) - 125(20) = -13750 < 0$. Hence by [4, Lemma 2.2]. [17, Lemma 3.3], the result follows.

Remark 3.1. According to [6], Ly is known to be a Hurwitz group making it to be $(2, 3, 7)$-generated and by [2] its symmetric genus is given by $1 + \frac{|G|}{84}$.

Proposition 3.2. Ly is

(i) not $(2A, 3A, 11)$-generated

(ii) $(2A, 3B, 11)$-generated.

Proof (i) Table 1 gives that $\zeta_{Ly}(2A, 3A, 11A) = 0 = \zeta_{Ly}(2A, 3A, 11B)$, proving (i).

(ii) By Table 1, we have that $\zeta_{Ly}(2A, 3B, 11A) = 6974 = \zeta_{Ly}(2A, 3B, 11B)$. The maximal subgroups of Ly containing elements of order 11 are $3{\cdot}McL{:}2; 2{\cdot}A_{11}$ and $3^5{:}(2 \times M_{11})$. We get that $11a$ and $11b$ of $3{\cdot}McL{:}2$ are each contained in one conjugate of $3{\cdot}McL{:}2$, $11a$ and $11b$ of $2{\cdot}A_{11}$ are each contained in three conjugates of $2{\cdot}A_{11}$, $11a$ and $11b$ of $3^5{:}(2 \times M_{11})$ are each contained in three conjugates of $3^5{:}(2 \times M_{11})$. By Tables 8, 10 and 12, we obtain that $\zeta_{3\cdot McL:2}(2a, 3d, 11a) = 11, \zeta_{2\cdot A_{11}}(2a, 3c, 11a) = 110, \zeta_{3^5:(2\times M_{11})}(2a, 3c, 11a) = 11$ and $\zeta_{3^5:(2\times M_{11})}(2a, 3d, 11a) = 22$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 11A) = 6974 - 11 - 3(110) - 3(11) - 3(22) = 6534$. By Tables 8, 10 and 12, we obtain that $\zeta_{3\cdot McL:2}(2a, 3d, 11b) = 11, \zeta_{2\cdot A_{11}}(2b, 3c, 11b) = 110, \zeta_{3^5:(2\times M_{11})}(2a, 3c, 11b) = 11$ and $\zeta_{3^5:(2\times M_{11})}(2a, 3d, 11b) = 20$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 11B) = 6974 - 11 - 3(110) - 3(11) - 3(20) = 6540$. Hence Ly is $(2A, 3B, 11)$-generated.

Proposition 3.3. Ly is

(i) not $(2A, 3A, 31)$-generated

(ii) $(2A, 3B, 31)$-generated.

Proof (i) By Table 1, we have that $\zeta_{Ly}(2A, 3A, 31A) = \zeta_{Ly}(2A, 3A, 31B) = 0 = \zeta_{Ly}(2A, 3A, 31C) = \zeta_{Ly}(2A, 3A, 31D) = \zeta_{Ly}(2A, 3A, 31E)$, proving (i).

(ii) By Table 1, we have $\zeta_{Ly}(2A, 3B, 31A) = \zeta_{Ly}(2A, 3B, 31B) = 7254 = \zeta_{Ly}(2A, 3B, 31C) = \zeta_{Ly}(2A, 3B, 31D) = \zeta_{Ly}(2A, 3B, 31E)$. The maximal subgroups of Ly containing elements of order 31 are $G_2(5)$ and $5^3{\cdot}L_3(5)$. We obtain that $31a$, $31b$, $31c$, $31d$ and $31e$ of $G_2(5)$ are each contained in one conjugate of $G_2(5)$ and the classes $31a$, $31b$, $31c$, $31d$, $31e$, $31f$, $31g$, $31h$, $31i$ and $31j$ of $5^3{\cdot}L_3(5)$ are each contained in one conjugate of $5^3{\cdot}L_3(5)$. By Tables 7 and 9, we obtain that $\zeta_{G_2(5)}(2a, 3b, 31a) = 496$ and $\zeta_{5^3\cdot L_3(5)}(2a, 3a, 31d) = 155 = \zeta_{5^3\cdot L_3(5)}(2a, 3a, 31h)$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 31A) = 7254-496-155-155 = 6448$, proving that Ly is $(2A, 3B, 31A)$-generated.

By Tables 7 and 9, we obtain that $\zeta_{G_2(5)}(2a, 3b, 31b) = 496$ and $\zeta_{5^3\cdot L_3(5)}(2a, 3a, 31i) = 155$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 31B) = 7254 - 496 - 155 = 6603$, proving that Ly is $(2A, 3B, 31B)$-generated. By Tables 7 and 9, we obtain that $\zeta_{G_2(5)}(2a, 3b, 31c) = 496$ and $\zeta_{5^3\cdot L_3(5)}(2a, 3a, 31b) = 155 = \zeta_{5^3\cdot L_3(5)}(2a, 3a, 31f) = \zeta_{5^3\cdot L_3(5)}(2a, 3a, 31j)$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 31C) = 7254-496-155-155-155 = 6293$, proving that Ly is $(2A, 3B, 31C)$-generated. Only $G_2(5)$ meets the $2A, 3B, 31D$ classes of Ly and so by Table 7 we obtain that $\zeta_{G_2(5)}(2a, 3b, 31d) = 496$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 31D) = 7254 - 496 = 6758$, proving that Ly is $(2A, 3B, 31D)$-generated.

The maximal subgroups $G_2(5)$ and $5^3{\cdot}L_3(5)$ meet the $2A$, $3B$, $31E$ classes of Ly. By Tables 7 and 9, we obtain that $\zeta_{G_2(5)}(2a, 3b, 31e) = 496$ and $\zeta_{5^3\cdot L_3(5)}(2a, 3a, 31a) = \zeta_{5^3\cdot L_3(5)}(2a, 3a, 31c) = 155 = \zeta_{5^3\cdot L_3(5)}(2a, 3a, 31e) = \zeta_{5^3\cdot L_3(5)}(2a, 3a, 31g)$. Thus we obtain that $\zeta^*_{Ly}(2A, 3B, 31E) = 7254 - 496 - 155 - 155 - 155 - 155 = 6138$, proving that Ly is $(2A, 3B, 31E)$-generated. Thus (ii) follows and the proof is complete.

Proposition 3.4. Ly is

(i) not $(2A, 3A, 37)$-generated

(ii) $(2A, 3B, 37)$-generated.

Proof (i) By Table 1, we obtain that $\zeta_{Ly}(2A, 3A, 37A) = 0 = \zeta_{Ly}(2A, 3A, 37B)$, proving (i).

(ii) By the same Table 1, we have that $\zeta_{Ly}(2A, 3B, 37A) = 7252 = \zeta_{Ly}(2A, 3B, 37B)$. None of the maximal subgroups of Ly contains elements of order 37 and therefore no contribution from any of them (maximal subgroups). We thus obtain that $\zeta^*_{Ly}(2A, 3B, 37A) = 7252 = \zeta^*_{Ly}(2A, 3B, 37B)$ and (ii) follows.

Proposition 3.5. Ly is

(i) not $(2A, 3A, 67)$-generated

(ii) $(2A, 3B, 67)$-generated.

Proof (i) Table 1 gives that $\zeta_{Ly}(2A, 3A, 67A) = 0 = \zeta_{Ly}(2A, 3A, 67B) = \zeta_{Ly}(2A, 3A, 67C)$, proving (i).

(ii) Again by Table 1, we have that $\zeta_{Ly}(2A, 3B, 67A) = 7705 = \zeta_{Ly}(2A, 3B, 67B) = \zeta_{Ly}(2A, 3B, 67C)$. None of the maximal subgroups of Ly contains elements of order 67 and so no contribution from any of them. We thus obtain that $\zeta^*_{Ly}(2A, 3B, 67A) = 7705 = \zeta^*_{Ly}(2A, 3B, 67B) = \zeta^*_{Ly}(2A, 3B, 67C)$. Hence (ii) follows and the proof is complete.

Table 1. Partial structure constants of Ly.

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(2A, 3A, tX)$	$\zeta_{Ly}(2A, 3B, tX)$
5A	2250000	0	0
5B	3750	0	6875
7A	168	0	8680
11A	66	0	6974
11B	66	0	6974
31A	31	0	7254
31B	31	0	7254
31C	31	0	7254
31D	31	0	7254
31E	31	0	7254
37A	37	0	7252
37B	37	0	7252
67A	67	0	7705
67B	67	0	7705
67C	67	0	7705

Table 2. Partial structure constants of $G_2(5)$.

tX	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(2a, 3a, tX)$	$\zeta_{G_2(5)}(2a, 3b, tX)$
5a	375000	0	0
5b	15000	0	0
5c	3750	0	625
5d	1250	0	625
5e	1250	0	625
7a	21	0	546
31a	31	0	496
31b	31	0	496
31c	31	0	496
31d	31	0	496
31e	31	0	496

Table 3. Partial structure constants of $3{\cdot}McL{:}2$.

tX	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(2a, 3a, tX)$	$\zeta_{3\cdot McL:2}(2a, 3b, tX)$	$\zeta_{3\cdot McL:2}(2a, 3c, tX)$	$\zeta_{3\cdot McL:2}(2a, 3d, tX)$
$5a$	4500	0	0	0	0
$5b$	150	0	0	0	125
$7a$	42	0	0	0	49
$11a$	66	0	0	0	11
$11b$	66	0	0	0	11
tX	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(2b, 3a, tX)$	$\zeta_{3\cdot McL:2}(2b, 3b, tX)$	$\zeta_{3\cdot McL:2}(2b, 3c, tX)$	$\zeta_{3\cdot McL:2}(2b, 3d, tX)$
$5a$	4500	0	0	0	0
$5b$	150	0	0	0	0
$7a$	42	0	0	0	0
$11a$	66	0	0	0	0
$11b$	66	0	0	0	0

Table 4. Partial structure constants of $5^3{\cdot}L_3(5)$.

tX	$\lvert C_{5^3\cdot L_3(5)}(tX)\rvert$	$\zeta_{5^3\cdot L_3(5)}(2a, 3a, tX)$
$5a$	375000	0
$5b$	250	0
$5c$	1250	0

$\mathbf{tX}$	$\lvert C_{5^3 \cdot L_3(5)}(tX)\rvert$	$\zeta_{5^3 \cdot L_3(5)}(2a,3a,tX)$
$5d$	1250	0
$31a$	31	155
$31b$	31	155
$31c$	31	155
$31d$	31	155
$31e$	31	155
$31f$	31	155
$31g$	31	155
$31h$	31	155
$31i$	31	155
$31j$	31	155

Table 5. *Partial structure constants of the* $2{\cdot}A_{11}$.

$\mathbf{tX}$	$\lvert C_{2A_{11}}(tX)\rvert$	$\zeta_{2\cdot A_{11}}(2a,3a,tX)$	$\zeta_{2\cdot A_{11}}(2a,3b,tX)$	$\zeta_{2\cdot A_{11}}(2a,3c,tX)$
$5a$	3600	0	0	0
$5b$	50	0	0	0
$7a$	168	0	0	0
$11a$	22	0	0	0
$11b$	22	0	0	0

$\mathbf{tX}$	$\lvert C_{2A_{11}}(tX)\rvert$	$\zeta_{2\cdot A_{11}}(2b,3a,tX)$	$\zeta_{2\cdot A_{11}}(2b,3b,tX)$	$\zeta_{2\cdot A_{11}}(2b,3c,tX)$
$5a$	3600	0	0	0
$5b$	50	0	25	100
$7a$	168	0	28	84
$11a$	22	0	0	110
$11b$	22	0	0	110

Table 6. *Partial structure constants of* $3^5{:}(2{\times}M_{11})$.

$\mathbf{tX}$	$\lvert C_{3^5:(2XM_{11})}(tX)\rvert$	$\zeta_{3^5:(2XM_{11})}(2a,3a,tX)$	$\zeta_{3^5:(2XM_{11})}(2a,3b,tX)$
$5a$	30	0	0
11a	22	0	0
$11b$	22	0	0

$\mathbf{tX}$	$\lvert C_{3^5:(2XM_{11})}(tX)\rvert$	$\zeta_{3^5:(2XM_{11})}(2a,3c,tX)$	$\zeta_{3^5:(2XM_{11})}(2a,3d,tX)$
$5a$	30	25	20
$11a$	22	11	22
$11b$	22	11	20

$\mathbf{tX}$	$\lvert C_{3^5:(2XM_{11})}(tX)\rvert$	$\zeta_{3^5:(2XM_{11})}(2b,3a,tX)$	$\zeta_{3^5:(2XM_{11})}(2b,3b,tX)$
$5a$	30	0	0
$11a$	22	0	0
$11b$	22	0	0

$\mathbf{tX}$	$\lvert C_{3^5:(2\times M_{11})}(tX)\rvert$	$\zeta_{3^5:(2\times M_{11})}(2b,3c,tX)$	$\zeta_{3^5:(2\times M_{11})}(2b,3d,tX)$
$5a$	30	0	0
$11a$	22	0	0
$11b$	22	0	0

$\mathbf{tX}$	$\lvert C_{3^5:(2\times M_{11})}(tX)\rvert$	$\zeta_{3^5:(2\times M_{11})}(2c,3a,tX)$	$\zeta_{3^5:(2\times M_{11})}(2c,3b,tX)$
$5a$	30	0	0
$11a$	22	0	0
$11b$	22	0	0

tX	$\lvert C_{3^5:(2\times M_{11})}(tX)\rvert$	$\zeta_{3^5:(2\times M_{11})}(2c,3c,tX)$	$\zeta_{3^5:(2\times M_{11})}(2c,3d,tX)$
$5a$	30	0	0
$11a$	22	0	0
$11b$	22	0	0

Tables 7 to 14 following here below, give the partial fusions into Ly of the relevant classes of its various maximal subgroups.

Table 7. *Partial fusions of* $G_2(5)$ *into* Ly.

$[g]_{Ly}$	2A	3A	3B	5A	5A	5B	5B	5B	7A	31A	31B	31C	31D	31E
↑														
$[h]_{G_2(5)}$	2a	3a	3b	5a	5b	5c	5d	5e	7a	31a	31b	31c	31d	31e

Table 8. *Partial fusions of* $3{\cdot}McL{:}2$ *into* Ly.

$[g]_{Ly}$	2A	2A	3A	3A	3B	3B	5A	5B	7A	11A	11B
↑											
$[h]_{3\cdot McL:2}$	2a	2b	3a	3b	3c	3d	5a	5b	7a	11a	11b

Table 9. *Partial fusions of* $5^3{\cdot}L_3(5)$ *into* Ly.

$[g]_{Ly}$	2A	2B	5A	5A	5B	5B	31E	31C
↑								
$[h]_{5^3\cdot L_3(5)}$	2a	3a	5a	5b	5c	5d	31a	31b
$[g]_{Ly}$	31E	31A	31E	31C	31E	31A	31B	31C
↑								
$[h]_{5^3\cdot L_3(5)}$	31c	31d	31e	31f	31g	31h	31i	31j

Table 10. *Partial fusions of* $5_+^{1+4}{:}4S_6$ *into* Ly.

$[g]_{Ly}$	2A	2A	3A	3B	3B	5A	5B	7A	11A	5B
↑										
$[h]_{2\cdot A_{11}}$	2a	2b	3a	3b	3c	5a	5b	7a	11a	11b

Table 11. *Partial fusions of* $5_+^{1+4}{:}4S_6$ *into* Ly.

$[g]_{Ly}$	2A	2A	3A	3B	5A	5A	5B	5A	5B	5B
↑										
$[h]_{5^{1+4}:4\cdot S_6}$	2a	2b	3a	3b	5a	5b	5c	5d	5e	5f

Table 12. *Partial fusions of* $3^5{:}(2\times M_{11})$ *into* Ly.

$[g]_{Ly}$	2A	2A	2A	3A	3B	3B	3B	5B	11A	11B
↑										
$[h]_{3^5:(2\times M_{11})}$	2a	2b	2c	3a	3b	3c	3d	5a	11a	11b

Table 13. *Partial fusions of* $3^{2+4}{:}2{\cdot}A_5{\cdot}D_8$ *into* Ly.

$[g]_{Ly}$	2A	2A	2A	3A	3B	3B	3A	3B	3B	5A
↑										
$[h]_{3^{2+4}:2\cdot A_5\cdot D_8}$	2a	2b	2c	3a	3b	3c	3d	3e	3f	5a

Table 14. Partial fusions of $37{:}18$ *into* Ly.

$[g]_{Ly}$	2A	3B	3B	37A	37B
$\uparrow$					
$[h]_{37:18}$	2a	3a	3b	37a	37b

3.2. The $(2,5,r)$-Generations of Ly

We shall study here the $(2,5,r)$-generations of Ly for all $r \in \{7,11,31,37,67\}$. The maximal subgroup $5^3{\cdot}L_3(5)$ contains elements of order 31 but will not have any contributions here since all its relevant structure constants are zero. The maximal subgroups having any contributions are $G_2(5); 3{\cdot}McL{:}2; 2{\cdot}A_{11}$ and $3^5{:}(2{\times}M_{11})$ because $G_2(5)$ contains elements of orders 7 and 31, $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ contain elements of orders 7 and 11, $3^5{:}(2{\times}M_{11})$ contains elements of order 11.

Proposition 3.6. Ly is $(2,5,7)$-generated.

Proof By Table 15, we have that $\zeta_{Ly}(2A,5A,7A) = 532$. The maximal subgroups of Ly containing elements of order 7 are $G_2(5); 3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$. We obtain that $7a$ of $G_2(5)$ is contained in eight conjugates of $G_2(5)$, $7a$ of $3{\cdot}McL{:}2$ is contained in four conjugates of $3{\cdot}McL{:}2$ and $7a$ of $2{\cdot}A_{11}$ is contained in one conjugate of $2{\cdot}A_{11}$. By Tables 7, 8 and 10, we obtain that $\zeta_{G_2(5)}(2a,5b,7a) = 21; \zeta_{3\cdot McL:2}(2a,5a,7a) = 7$ and $\zeta_{2\cdot A_{11}}(2b,5a,7a) = 28$. Thus we obtain that $\zeta^*_{Ly}(2A,5A,7A) = 532-8(21)-4(7)-1(28) = 308$, proving that Ly is $(2A,5A,7A)$-generated.

According to Table 15, we have that $\zeta_{Ly}(2A,5B,7A) = 312200$. By Table 7, we obtain that $\zeta_{G_2(5)}(2a,5c,7a) = 28; \zeta_{G_2(5)}(2a,5d,7a) = 287$ and $\zeta_{G_2(5)}(2a,5e,7a) = 315$ and by Tables 8 and 10, we obtain that $\zeta_{3\cdot McL:2}(2a,5b,7a) = 476$ and $\zeta_{2\cdot A_{11}}(2b,5b,7a) = 504$. Thus we obtain that $\zeta^*_{Ly}(2A,5B,7A) = 312200-8(28)-8(287)-8(315)-4(476)-1(504) = 304752$, proving that Ly is $(2A,5B,7A)$-generated. Hence the result follows and the proof is complete.

Proposition 3.7. Ly is $(2,5,11)$-generated.

Proof By Table 15, we have that $\zeta_{Ly}(2A,5A,11A) = 396 = \zeta_{Ly}(2A,5A,11B)$. The maximal subgroups of Ly containing elements of order 11 are $3{\cdot}McL{:}2; 2{\cdot}A_{11}$ and $3^5{:}(2{\times}M_{11})$. However none of them has a contribution because $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ have their relevant structure constants all zero and $3^5{:}(2{\times}M_{11})$ does not meet $2A,5A,11A,11B$ classes of Ly. Therefore $\zeta^*_{Ly}(2A,5A,11A) = 396 = \zeta^*_{Ly}(2A,5A,11B)$ proving that Ly is $(2A,5A,11)$-generated.

By Table 15, we have $\zeta_{Ly}(2A,5B,11A) = 355960 = \zeta_{Ly}(2A,5B,11B)$. We obtain that $11a$ and $11b$ of $3{\cdot}McL{:}2$ are each contained in one conjugate of $3{\cdot}McL{:}2, 11a$ and $11b$ of $2{\cdot}A_{11}$ are each contained in three conjugates of $2{\cdot}A_{11}, 11a$ and $11b$ of $3^5{:}(2{\times}M_{11})$ are each contained in three conjugates of $3^5{:}(2{\times}M_{11})$. By Tables 8, 10 and 12, we have that $\zeta_{3\cdot McL:2}(2a,5b,11a) = 187 = \zeta_{3\cdot McL:2}(2a,5b,11b), \zeta_{2\cdot A_{11}}(2b,5b,11a) = 660 = \zeta_{2\cdot A_{11}}(2b,5b,11b)$ and $\zeta_{3^5:(2\times M_{11})}(2a,5a,11a) = 99 = \zeta_{3^5:(2\times M_{11})}(2a,5a,11b)$. We thus obtain that $\zeta^*_{Ly}(2A,5B,11A) = 355960-1(187)-3(660)-3(99) = 353496 = \zeta^*_{Ly}(2A,5B,11B)$, proving that Ly is $(2A,5B,11)$-generated. Hence the result follows.

Proposition 3.8. Ly is $(2,5,31)$-generated.

Proof By Table 15, we have that $\zeta_{Ly}(2A,5A,31A) = \zeta_{Ly}(2A,5A,31B) = 589 = \zeta_{Ly}(2A,5A,31C) = \zeta_{Ly}(2A,5A,31D) = \zeta_{Ly}(2A,5A,31E)$. Only one maximal subgroup of Ly contains elements of order 31 viz. $G_2(5)$ and that $31a,31b,31c,31d,31e$ of $G_2(5)$ are each contained in one conjugate of $G_2(5)$. By Table 7, we have that $\zeta_{G_2(5)}(2a,5b,31a) = \zeta_{G_2(5)}(2a,5b,31b) = 31 = \zeta_{G_2(5)}(2a,5b,31c) = \zeta_{G_2(5)}(2a,5b,31d) = \zeta_{G_2(5)}(2a,5b,31e)$. Thus we obtain that $\zeta^*_{Ly}(2A,5A,31A) = 589-31 = 558 = \zeta^*_{Ly}(2A,5A,31B) = \zeta^*_{Ly}(2A,5A,31C) = \zeta^*_{Ly}(2A,5A,31D) = \zeta^*_{Ly}(2A,5A,31E)$.

By Table 15, we have that $\zeta_{Ly}(2A,5B,31A) = \zeta_{Ly}(2A,5B,31B) = 346425 = \zeta_{Ly}(2A,5B,31C) = \zeta_{Ly}(2A,5B,31D) = \zeta_{Ly}(2A,5B,31E)$. By Table 7, we have that $\zeta_{G_2(5)}(2a,5c,31a) = \zeta_{G_2(5)}(2a,5c,31b) = 93 = \zeta_{G_2(5)}(2a,5c,31c) = \zeta_{G_2(5)}(2a,5c,31d) = \zeta_{G_2(5)}(2a,5c,31e), \zeta_{G_2(5)}(2a,5d,31a) = \zeta_{G_2(5)}(2a,5d,31b) = 217 = \zeta_{G_2(5)}(2a,5d,31c) = \zeta_{G_2(5)}(2a,5d,31d) = \zeta_{G_2(5)}(2a,5d,31e), \zeta_{G_2(5)}(2a,5e,31a) = \zeta_{G_2(5)}(2a,5e,31b) = 310 = \zeta_{G_2(5)}(2a,5e,31c) = \zeta_{G_2(5)}(2a,5e,31d) = \zeta_{G_2(5)}(2a,5e,31e)$. We obtain that $\zeta^*_{Ly}(2A,5B,31A) = 346425-93-217-310 = 345805 = \zeta^*_{Ly}(2A,5B,31B) = \zeta^*_{Ly}(2A,5B,31C) = \zeta^*_{Ly}(2A,5B,31D) = \zeta^*_{Ly}(2A,5B,31E)$. Hence Ly is $(2,5,31)$-generated.

Proposition 3.9. Ly is $(2,5,37)$-generated.

Proof By Table 15, we have that $\zeta_{Ly}(2A,5A,37A) = 629 = \zeta_{Ly}(2A,5A,37B)$ and $\zeta_{Ly}(2A,5B,37A) = 330595 = \zeta_{Ly}(2A,5B,37B)$. There is no contribution from any of the maximal subgroups of Ly because none contains elements of order 37. We thus obtain that $\zeta^*_{Ly}(2A,5A,37A) = 629 = \zeta^*_{Ly}(2A,5A,37B)$ and $\zeta^*_{Ly}(2A,5B,37A) = 330595 = \zeta^*_{Ly}(2A,5B,37B)$ proving that Ly is $(2,5,37)$-generated.

Proposition 3.10. Ly is $(2,5,67)$-generated.

Proof By Table 15, we have that $\zeta_{Ly}(2A,5A,67A) = 871 = \zeta_{Ly}(2A,5A,67B) = \zeta_{Ly}(2A,5A,67C)$ and $\zeta_{Ly}(2A,5B,67A) = 320930 = \zeta_{Ly}(2A,5B,67B) = \zeta_{Ly}(2A,5B,67C)$. There is no contribution from any of the maximal subgroups of Ly because none contains elements of order 67. We thus obtain that $\zeta^*_{Ly}(2A,5A,67A) = 871 = \zeta^*_{Ly}(2A,5A,67B) = \zeta^*_{Ly}(2A,5A,67C)$ and $\zeta^*_{Ly}(2A,5B,67A) = 320930 = \zeta^*_{Ly}(2A,5B,67B) = \zeta^*_{Ly}(2A,5B,67C)$ proving that Ly is $(2,5,67)$- generated.

Table 15. *Partial structure constants of* Ly.

$\mathbf{tX}$	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(2A,5A,tX)$	$\zeta_{Ly}(2A,5B,tX)$
$7A$	168	532	312200
$11A$	66	396	355960
$11B$	66	396	355960
$31A$	31	589	346425
$31B$	31	589	346425
$31C$	31	589	346425
$31D$	31	589	346425
$31E$	31	589	346425
$37A$	37	629	330595
$37B$	37	629	330595
$67A$	67	871	320930
$67B$	67	871	320930
$67C$	67	871	320930

Table 16. *Partial structure constants of* $G_2(5)$.

$\mathbf{tX}$	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(2a,5a,tX)$	$\zeta_{G_2(5)}(2a,5b,tX)$	$\zeta_{G_2(5)}(2a,5c,tX)$
$7a$	21	0	21	28
$31a$	31	0	31	93
$31b$	31	0	31	93
$31c$	31	0	31	93
$31d$	31	0	31	93
$31e$	31	0	31	93

$\mathbf{tX}$	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(2a,5d,tX)$	$\zeta_{G_2(5)}(2a,5e,tX)$
$7a$	21	287	315
$31a$	31	217	310
$31b$	31	217	310
$31c$	31	217	310
$31d$	31	217	310
$31e$	31	217	310

Table 17. *Partial structure constants of* $3{\cdot}McL{:}2$.

$\mathbf{tX}$	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(2a,5a,tX)$	$\zeta_{3\cdot McL:2}(2a,5b,tX)$
$7a$	42	7	476
$11a$	66	0	187
$11b$	66	0	187

$\mathbf{tX}$	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(2b,5a,tX)$	$\zeta_{3\cdot McL:2}(2b,5b,tX)$
$7a$	42	0	0
$11a$	66	0	0
$11b$	66	0	0

Table 18. *Partial structure constants of* $2{\cdot}A_{11}$.

$\mathbf{tX}$	$\lvert C_{2\cdot A_{11}}(tX)\rvert$	$\zeta_{2\cdot A_{11}}(2a,5a,tX)$	$\zeta_{2\cdot A_{11}}(2a,5b,tX)$	$\zeta_{2\cdot A_{11}}(2b,5a,tX)$	$\zeta_{2\cdot A_{11}}(2b,5b,tX)$
$7a$	168	0	0	28	504
$11a$	22	0	0	0	660
$11b$	22	0	0	0	660

Table 19. Partial structure constants of $3^5{:}(2 \times M_{11})$.

tX	$\lvert C_{3^5:(2\times M_{11})}(tX)\rvert$	$\zeta_{3^5:(2\times M_{11})}(2a,5a,tX)$
$11a$	22	99
$11b$	22	99

3.3. The $(2,7,r)$-Generations of Ly

For the $(2,7,r)$-generations of Ly, we shall consider all $r \in \{11,31,37,67\}$. The maximal subgroups having any contributions are $G_2(5); 3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ because $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ contain elements of order 11, while $G_2(5)$ contains elements of order 31.

Proposition 3.11. Ly is $(2,7,11)$-generated.

Proof By Table 20, we have that $\zeta_{Ly}(2A,7A,11A) = 7804071 = \zeta_{Ly}(2A,7A,11B)$. The maximal subgroups of Ly containing elements of order 11 are $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$. We have that $11a, 11b$ of $3{\cdot}McL{:}2$ are each contained in one conjugate of $3{\cdot}McL{:}2$ while $11a, 11b$ of $2{\cdot}A_{11}$ are each contained in three conjugates of $2{\cdot}A_{11}$. By Tables 8 and 10, we obtain that $\zeta_{3\cdot McL:2}(2a,7a,11a) = 1056 = \zeta_{3\cdot McL:2}(2a,7a,11b)$ and $\zeta_{2\cdot A_{11}}(2b,7a,11a) = 55 = \zeta_{2\cdot A_{11}}(2b,7a,11b)$. Thus $\zeta^*_{Ly}(2A,7A,11A) = 7804071 - 1(1056) - 3(55) = 7802850 = \zeta^*_{Ly}(2A,7A,11B)$, proving that Ly is $(2,7,11)$-generated.

Proposition 3.12. Ly is $(2,7,31)$-generated.

Proof By Table 20, we have that $\zeta_{Ly}(2A,7A,31A) = \zeta_{Ly}(2A,7A,31B) = 7719372 = \zeta_{Ly}(2A,7A,31C) = \zeta_{Ly}(2A,7A,31D) = \zeta_{Ly}(2A,7A,31E)$. There is only one maximal subgroup of Ly containing elements of order 31 viz. $G_2(5)$. We obtain that $31a, 31b, 31c, 31d, 31e$ of $G_2(5)$ are each contained in one conjugate of $G_2(5)$. By Table 7, we have that $\zeta_{G_2(5)}(2a,7a,31a) = \zeta_{G_2(5)}(2a,7a,31b) = 19406 = \zeta_{G_2(5)}(2a,7a,31c) = \zeta_{G_2(5)}(2a,7a,31d) = \zeta_{G_2(5)}(2a,7a,31e)$. Thus $\zeta^*_{Ly}(2A,7A,31A) = 7719372 - 19406 = 7699966 = \zeta^*_{Ly}(2A,7A,31B) = \zeta^*_{Ly}(2A,7A,31C) = \zeta^*_{Ly}(2A,7A,31D) = \zeta^*_{Ly}(2A,7A,31E)$, thus proving that Ly is $(2,7,31)$-generated.

Table 20. Partial structure constants of Ly.

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(2A,7A,tX)$
$11A$	66	7804071
$11B$	66	7804071
$31A$	31	7719372
$31B$	31	7719372
$31C$	31	7719372
$31D$	31	7719372
$31E$	31	7719372
$37A$	37	7560025
$37B$	37	7560025
$67A$	67	7511102
$67B$	67	7511102
$67C$	67	7511102

Proposition 3.13. Ly is $(2,7,37)$-generated.

Proof By Table 20, we have that $\zeta_{Ly}(2A,7A,37A) = 7560025 = \zeta_{Ly}(2A,7A,37B)$. There is no contribution from any maximal subgroup because none contains elements of order 37. Thus we obtain that $\zeta^*_{Ly}(2A,7A,37A) = 7560025 = \zeta^*_{Ly}(2A,7A,37B)$ proving that Ly is $(2,7,37)$-generated.

Proposition 3.14. Ly is $(2,7,67)$-generated.

Proof By Table 20, we have that $\zeta_{Ly}(2A,7A,67A) = 7511102 = \zeta_{Ly}(2A,7A,67B) = \zeta_{Ly}(2A,7A,67C)$. There is no contribution from any maximal subgroup because none contains elements of order 67. Thus we obtain that $\zeta^*_{Ly}(2A,7A,67A) = 7511102 = \zeta^*_{Ly}(2A,7A,67B) = \zeta^*_{Ly}(2A,7A,67C)$ proving that Ly is $(2,7,67)$-generated.

Table 21. Partial structure constants of $G_2(5)$.

tX	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(2a,7a,tX)$
$31a$	31	19406
$31b$	31	19406
$31c$	31	19406
$31d$	31	19406
$31e$	31	19406

Table 22. Partial structure constants of $3{\cdot}McL{:}2$.

tX	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(2a,7a,tX)$	$\zeta_{3\cdot McL:2}(2b,7a,tX)$
$11a$	66	1056	0
$11b$	66	1056	0

Table 23. Partial structure constants of $2{\cdot}A_{11}$.

tX	$\lvert C_{2\cdot A_{11}}(tX)\rvert$	$\zeta_{2\cdot A_{11}}(2a,7a,tX)$	$\zeta_{2\cdot A_{11}}(2b,7a,tX)$
$11a$	22	0	55
$11b$	22	0	55

3.4. The $(2, 11, r)$-Generations of Ly

We shall consider all $r \in \{31, 37, 67\}$. No maximal subgroup has any contribution here because none contains elements of order 31 and 37.

However 67:22 is the only one containing elements of order 67 but will not have any contribution since all its relevant structure constants are zero.

Proposition 3.15. Ly is $(2, 11, 31)$-generated.

Proof By Table 24, we have that $\zeta_{Ly}(2A, 11A, 31A) = \zeta_{Ly}(2A, 11A, 31B) = 19645413 = \zeta_{Ly}(2A, 11A, 31C) = \zeta_{Ly}(2A, 11A, 31D) = \zeta_{Ly}(2A, 11A, 31E)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(2A, 11A, 31A) = \zeta^*_{Ly}(2A, 11A, 31B) = 19645413 = \zeta^*_{Ly}(2A, 11A, 31C) = \zeta^*_{Ly}(2A, 11A, 31D) = \zeta^*_{Ly}(2A, 11A, 31E)$ proving that Ly is $(2, 11, 31)$-generated.

Proposition 3.16. Ly is $(2, 11, 37)$-generated.

Proof By Table 24, we have that $\zeta_{Ly}(2A, 11A, 37A) = \zeta_{Ly}(2A, 11A, 37B) = 19767583 = \zeta_{Ly}(2A, 11B, 37A) = \zeta_{Ly}(2A, 11B, 37B)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(2A, 11A, 37A) = \zeta^*_{Ly}(2A, 11A, 37B) = 19767583 = \zeta^*_{Ly}(2A, 11B, 37A) = \zeta^*_{Ly}(2A, 11B, 37B)$ proving that Ly is $(2, 11, 37)$-generated.

Proposition 3.17. Ly is $(2, 11, 67)$-generated.

Proof By Table 24, we have that $\zeta_{Ly}(2A, 11A, 67A) = \zeta_{Ly}(2A, 11A, 67B) = \zeta_{Ly}(2A, 11A, 67C) = 19724130 = \zeta_{Ly}(2A, 11B, 67A) = \zeta_{Ly}(2A, 11B, 67B) = \zeta_{Ly}(2A, 11B, 67C)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(2A, 11A, 67A) = \zeta^*_{Ly}(2A, 11A, 67B) = \zeta^*_{Ly}(2A, 11A, 67C) = 19724130 = \zeta^*_{Ly}(2A, 11B, 67A) = \zeta^*_{Ly}(2A, 11B, 67B) = \zeta^*_{Ly}(2A, 11B, 67C)$ proving that Ly is $(2, 11, 67)$-generated.

Table 24. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(2A, 11A, tX)$	$\zeta_{Ly}(2A, 11B, tX)$
$31A$	31	19645413	19645413
$31B$	31	19645413	19645413
$31C$	31	19645413	19645413
$31D$	31	19645413	19645413
$31E$	31	19645413	19645413
$37A$	37	19767583	19767583
$37B$	37	19767583	19767583
$67A$	67	19724130	19724130
$67B$	67	19724130	19724130
$67C$	67	19724130	19724130

Table 25. *Partial fusions of* 67:22 *into* Ly.

$[g]_{Ly}$	2A	11A	11B	11A	11A	11A	11B	11B	11B	11A	11B	67A	67B	67C
$\uparrow$														
$[h]_{67:22}$	$2a$	$11a$	$11b$	$11c$	$11d$	$11e$	$11f$	$11g$	$11h$	$11i$	$11j$	$67a$	$67b$	$67c$

3.5. The $(2, 31, r)$-Generations of Ly

In this case, we shall consider all $r \in \{37, 67\}$. No maximal subgroup has any contribution here since there is none containing elements of orders 37 and 67.

Proposition 3.18. Ly is $(2, 31, 37)$-generated.

Proof By Table 26, we have that $\zeta_{Ly}(2A, 31A, 37A) = \zeta_{Ly}(2A, 31B, 37A) = \zeta_{Ly}(2A, 31C, 37A) = \zeta_{Ly}(2A, 31D, 37A) = \zeta_{Ly}(2A, 31E, 37A) = 41833125 = \zeta_{Ly}(2A, 31A, 37B) = \zeta_{Ly}(2A, 31B, 37B) = \zeta_{Ly}(2A, 31C, 37B) = \zeta_{Ly}(2A, 31D, 37B) = \zeta_{Ly}(2A, 31E, 37B)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(2A, 31A, 37A) = \zeta^*_{Ly}(2A, 31B, 37A) = \zeta^*_{Ly}(2A, 31C, 37A) = \zeta^*_{Ly}(2A, 31D, 37A) = \zeta^*_{Ly}(2A, 31E, 37A) = 41833125 = \zeta^*_{Ly}(2A, 31A, 37B) = \zeta^*_{Ly}(2A, 31B, 37B) = \zeta^*_{Ly}(2A, 31C, 37B) = \zeta^*_{Ly}(2A, 31D, 37B) = \zeta^*_{Ly}(2A, 31E, 37B)$ proving that Ly is $(2, 31, 37)$-generated.

Proposition 3.19. Ly is $(2, 31, 67)$-generated.

Proof By Table 26, we have that $\zeta_{Ly}(2A, 31A, 67A) = \zeta_{Ly}(2A, 31B, 67A) = \zeta_{Ly}(2A, 31C, 67A) = \zeta_{Ly}(2A, 31D, 67A) = \zeta_{Ly}(2A, 31E, 67A) = \zeta_{Ly}(2A, 31A, 67B) = \zeta_{Ly}(2A, 31B, 67B) = 41833125 = \zeta_{Ly}(2A, 31C, 67B) = \zeta_{Ly}(2A, 31D, 67B) = \zeta_{Ly}(2A, 31E, 67B) = \zeta_{Ly}(2A, 31A, 67C) = \zeta_{Ly}(2A, 31B, 67C) = \zeta_{Ly}(2A, 31C, 67C) = \zeta_{Ly}(2A, 31D, 67C) = \zeta_{Ly}(2A, 31E, 67C)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(2A, 31A, 67A) = \zeta^*_{Ly}(2A, 31B, 67A) = \zeta^*_{Ly}(2A, 31C, 67A) = \zeta^*_{Ly}(2A, 31D, 67A) = \zeta^*_{Ly}(2A, 31E, 67A) = \zeta^*_{Ly}(2A, 31A, 67B) = \zeta^*_{Ly}(2A, 31B, 67B) = 41833125 = \zeta^*_{Ly}(2A, 31C, 67B) = \zeta^*_{Ly}(2A, 31D, 67B) = \zeta^*_{Ly}(2A, 31E, 67B) = \zeta^*_{Ly}(2A, 31A, 67C) = \zeta^*_{Ly}(2A, 31B, 67C) = \zeta^*_{Ly}(2A, 31C, 67C) = \zeta^*_{Ly}(2A, 31D, 67C) = \zeta^*_{Ly}(2A, 31E, 67C)$ proving that Ly is $(2, 31, 67)$-generated.

Table 26. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(2A,31A,tX)$	$\zeta_{Ly}(2A,31B,tX)$	$\zeta_{Ly}(2A,31C,tX)$
37A	37	41833125	41833125	41833125
37B	37	41833125	41833125	41833125
67A	67	41833125	41833125	41833125
67B	67	41833125	41833125	41833125
67C	67	41833125	41833125	41833125

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(2A,31D,tX)$	$\zeta_{Ly}(2A,31E,tX)$
37A	37	41833125	41833125
37B	37	41833125	41833125
67A	67	41833125	41833125
67B	67	41833125	41833125
67C	67	41833125	41833125

3.6. The $(2,37,r)$-Generations of Ly

We shall consider only $r = 67$. No maximal subgroup has any contribution here since there is none containing elements of order 67.

Proposition 3.20. Ly is $(2,37,67)$-generated.

Proof By Table 27, we have that $\zeta_{Ly}(2A,37A,67A) = \zeta_{Ly}(2A,37A,67B) = \zeta_{Ly}(2A,37A,67C) = 34610391 = \zeta_{Ly}(2A,37B,67A) = \zeta_{Ly}(2A,37B,67B) = \zeta_{Ly}(2A,37B,67C)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(2A,37A,67A) = \zeta^*_{Ly}(2A,37A,67B) = \zeta^*_{Ly}(2A,37A,67C) = 34610391 = \zeta^*_{Ly}(2A,37B,67A) = \zeta^*_{Ly}(2A,37B,67B) = \zeta^*_{Ly}(2A,37B,67C)$ proving that Ly is $(2,37,67)$-generated.

Table 27. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(2A,37A,tX)$	$\zeta_{Ly}(2A,37B,tX)$
$67A$	67	34610391	34610391
$67B$	67	34610391	34610391
$67C$	67	34610391	34610391

4. The $(3,q,r)$-Generations of Ly

We consider here all $q \in \{5,7,11,31,37\}$ and all $r \in \{7,11,31,37,67\}$.

4.1. The $(3,5,r)$-Generations of Ly

We shall consider all $r \in \{7,11,31,37,67\}$. The maximal subgroups having any contributions are $G_2(5); 3{\cdot}McL{:}2; 5^3{\cdot}L_3(5); 2{\cdot}A_{11}; 3^5{:}(2\times M_{11})$ because $G_2(5)$ contains elements of orders 7 and 31, $3{\cdot}McL{:}2; 2{\cdot}A_{11}$ contain elements of orders 7 and 11, $5^3{\cdot}L_3(5)$ contains elements of order 31 and $3^5{:}(2\times M_{11})$ contains elements of order 11.

Proposition 4.1. Ly is

(i) not $(3A,5A,7A)$-generated

(ii) $(3A,5B,7A),(3B,5A,7A),(3B,5B,7A)$-generated.

Proof (i) By Table 28 above, we have that $\zeta_{Ly}(3A,5A,7A) = 7 < 168 = |C_{Ly}(7A)|$, proving (i).

(ii) The maximal subgroups of Ly containing elements of order 7 are $G_2(5); 3{\cdot}McL{:}2; 2{\cdot}A_{11}$. We get that $7a$ of $G_2(5)$ is contained in eight conjugates of $G_2(5)$, $7a$ of $3{\cdot}McL{:}2$ is contained in four conjugates of $3{\cdot}McL{:}2$ and $7a$ of $2{\cdot}A_{11}$ is contained in one conjugate of $2{\cdot}A_{11}$. By Table 28, we have that $\zeta_{Ly}(3A,5B,7A) = 10528$. Only $G_2(5)$ meets the $3A,5B,7A$ classes of Ly and by Table 7, we have that $\zeta_{G_2(5)}(3a,5c,7a) = 21$ and $\zeta_{G_2(5)}(3a,5e,7a) = 11$. We obtain that $\zeta^*_{Ly}(3A,5B,7A) = 10528 - 8(21) - 8(11) = 10272$, proving that Ly is $(3A,5B,7A)$-generated.

By Table 28, we have that $\zeta_{Ly}(3B,5A,7A) = 148197$. By Tables 7, 8 and 10, we have that $\zeta_{G_2(5)}(3b,5a,7a) = 21, \zeta_{G_2(5)}(3b,5b,7a) = 525, \zeta_{3\cdot McL:2}(3c,5a,7a) = 56$ and $\zeta_{3\cdot McL:2}(3d,5a,7a) = 595, \zeta_{2A_{11}}(3b,5a,7a) = 21, \zeta_{2A_{11}}(3c,5a,7a) = 168$. We thus obtain that $\zeta^*_{Ly}(3B,5A,7A) = 148197 - 8(21) - 8(525) - 4(56) - 4(595) - 1(21) - 1(168) = 141036$, proving that Ly is (3B,5A,7A)-generated.

From Table 28, we have that $\zeta_{Ly}(3B,5B,7A) = 76620040$. By Tables 7, 8, 10, we have that $\zeta_{G_2(5)}(3b,5c,7a) = 1925, \zeta_{G_2(5)}(3b,5d,7a) = 4900$ and $\zeta_{G_2(5)}(3b,5e,7a) = 6300, \zeta_{3\cdot McL:2}(3c,5b,7a) = 364$ and $\zeta_{3\cdot McL:2}(3d,5b,7a) = 50400, \zeta_{2\cdot A_{11}}(3b,5b,7a) = 504$ and $\zeta_{2\cdot A_{11}}(3c,5b,7a) = 1680$. We thus obtain that $\zeta^*_{Ly}(3B,5B,7A) = 76620040 - 8(1925) - 8(4900) - 8(6300) - 4(364) - 4(50400) - 1(504) - 1(1680) = 76309800$, proving that Ly is $(3B,5B,7A)$-generated and the proof is complete.

Proposition 4.2. Ly is

(i) not $(3A,5A,11A),(3A,5A,11B)$-generated

(ii) $(3A,5B,11),(3B,5A,11),(3B,5B,11)$-generated

Proof (i) From Table 28, we have that $\zeta_{Ly}(3A, 5A, 11A) = 22 = \zeta_{Ly}(3A, 5A, 11B)$. The maximal subgroups containing elements of order 11 are $3{\cdot}McL{:}2; 2{\cdot}A_{11}; 3^5{:}(2{\times}M_{11})$. The classes $11a, 11b$ of $3{\cdot}McL{:}2$ are each contained in one conjugate of $3{\cdot}McL{:}2$, $11a, 11b$ of $2{\cdot}A_{11}$ are each contained in three conjugates of $2{\cdot}A_{11}$, $11a, 11b$ of $3^5{:}(2{\times}M_{11})$ are each contained in three conjugates of $3^5{:}(2{\times}M_{11})$. Although $3{\cdot}McL{:}2; 2{\cdot}A_{11}$ meet the $3A, 5A, 11A, 11B$ classes of Ly, only $3{\cdot}McL{:}2$ has a contribution since $2{\cdot}A_{11}$ has its relevant structure constants all zero. By Table 8, we obtain that $\zeta_{3{\cdot}McL:2}(3b, 5a, 11a) = 22 = \zeta_{3{\cdot}McL:2}(3b, 5a, 11b)$ so that $\zeta^*_{Ly}(3A, 5A, 11A) = 22 - 22 = 0 = \zeta^*_{Ly}(3A, 5A, 11B)$, thus proving (i).

(ii) By Table 28, we have that $\zeta_{Ly}(3A, 5B, 11A) = 5071 = \zeta_{Ly}(3A, 5B, 11B)$. Only $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ meet the $3A$, $5B$, $11A$, $11B$ classes of Ly and so by Tables 8 and 10, we have that $\zeta_{3{\cdot}McL:2}(3b, 5b, 11a) = 418 = \zeta_{3{\cdot}McL:2}(3b, 5b, 11b)$ and $\zeta_{2{\cdot}A_{11}}(3a, 5b, 11a) = 11 = \zeta_{2{\cdot}A_{11}}(3a, 5b, 11b)$. Thus we obtain that $\zeta^*_{Ly}(3A, 5B, 11A) = 5071 - 1(418) - 3(11) = 4620 = \zeta^*_{Ly}(3A, 5B, 11B)$, proving that Ly is $(3A, 5B, 11)$-generated.

By Table 28, we have that $\zeta_{Ly}(3B, 5A, 11A) = 131824 = \zeta_{Ly}(3B, 5A, 11B)$. Although $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ meet the $3B, 5A, 11A, 11B$ classes of Ly, only $3{\cdot}McL{:}2$ has a contribution since $2{\cdot}A_{11}$ has its relevant structure constants all zero. By Table 8, we have that $\zeta_{3{\cdot}McL:2}(3c, 5a, 11a) = 22 = \zeta_{3{\cdot}McL:2}(3c, 5a, 11b)$ and $\zeta_{3{\cdot}McL:2}(3d, 5a, 11a) = 1122 = \zeta_{3{\cdot}McL:2}(3d, 5a, 11b)$. We then get that $\zeta^*_{Ly}(3B, 5A, 11A) = 131824 - 22 - 1122 = 130680 = \zeta^*_{Ly}(3B, 5A, 11B)$, proving that Ly is $(3B, 5A, 11)$-generated.

From Table 28, we have that $\zeta_{Ly}(3B, 5B, 11A) = 80152600 = \zeta_{Ly}(3B, 5B, 11B)$. From Table 8, we obtain that $\zeta_{3{\cdot}McL:2}(3c, 5b, 11a) = 682 = \zeta_{3{\cdot}McL:2}(3c, 5b, 11b)$ and $\zeta_{3{\cdot}McL:2}(3d, 5b, 11a) = 34485 = \zeta_{3{\cdot}McL:2}(3d, 5b, 11b)$. By Table 10, we have that $\zeta_{2{\cdot}A_{11}}(3b, 5b, 11a) = 242 = \zeta_{2{\cdot}A_{11}}(3b, 5b, 11b)$ and $\zeta_{2{\cdot}A_{11}}(3c, 5b, 11a) = 2816 = \zeta_{2{\cdot}A_{11}}(3c, 5b, 11b)$.

From Table 12, we obtain that $\zeta_{3^5:(2\times M_{11})}(3c, 5a, 11a) = 891 = \zeta_{3^5:(2\times M_{11})}(3c, 5a, 11b)$ and $\zeta_{3^5:(2\times M_{11})}(3d, 5a, 11a) = 1782 = \zeta_{3^5:(2\times M_{11})}(3d, 5a, 11b)$. Thus we obtain that $\zeta^*_{Ly}(3B, 5B, 11A) = 80152600 - 1(682) - 1(34485) - 3(242) - 3(2816) - 3(891) - 3(1782) = 80100240 = \zeta^*_{Ly}(3B, 5B, 11B)$, proving that Ly is $(3B, 5B, 11)$-generated. Hence the proof is complete.

Proposition 4.3. Ly is

(i) not $(3A, 5A, 31)$-generated

(ii) $(3A, 5B, 31), (3B, 5A, 31), (3B, 5B, 31)$-generated

Proof (i) By Table 28, we have that $\zeta_{Ly}(3A, 5A, 31A) = \zeta_{Ly}(3A, 5A, 31B) = 0 = \zeta_{Ly}(3A, 5A, 31C) = \zeta_{Ly}(3A, 5A, 31D) = \zeta_{Ly}(3A, 5A, 31E)$, proving (i).

(ii) The maximal subgroups containing elements of order 31 are $G_2(5); 5^3{\cdot}L_3(5)$. Thus $31a, 31b, 31c, 31d, 31e$ of $G_2(5)$ are each contained in one conjugate of $G_2(5)$. The classes $31a, 31b, 31c, 31d, 31e, 31f, 31g, 31h, 31i, 31j$ of $5^3{\cdot}L_3(5)$ are each contained in one conjugate of $5^3{\cdot}L_3(5)$. By Table 28, we have that $\zeta_{Ly}(3A, 5B, 31A) = \zeta_{Ly}(3A, 5B, 31B) = 4030 = \zeta_{Ly}(3A, 5B, 31C) = \zeta_{Ly}(3A, 5B, 31D) = \zeta_{Ly}(3A, 5B, 31E)$. Only $G_2(5)$ meets $3A, 5B, 31A, 31B, 31C, 31D, 31E$ classes of Ly. Using Table 7, we have that $\zeta_{G_2(5)}(3a, 5e, 31a) = \zeta_{G_2(5)}(3a, 5e, 31b) = 31 = \zeta_{G_2(5)}(3a, 5e, 31c) = \zeta_{G_2(5)}(3a, 5e, 31d) = \zeta_{G_2(5)}(3a, 5e, 31e)$. Thus we obtain that $\zeta^*_{Ly}(3A, 5B, 31A) = 4030 - 31 = 3999 = \zeta^*_{Ly}(3A, 5B, 31B) = \zeta^*_{Ly}(3A, 5B, 31C) = \zeta^*_{Ly}(3A, 5B, 31D) = \zeta^*_{Ly}(3A, 5B, 31E)$. Hence Ly is $(3A, 5B, 31)$-generated.

By Table 28, we have that $\zeta_{Ly}(3B, 5A, 31A) = \zeta_{Ly}(3B, 5A, 31B) = 124651 = \zeta_{Ly}(3B, 5A, 31C) = \zeta_{Ly}(3B, 5A, 31D) = \zeta_{Ly}(3B, 5A, 31E)$. By Table 7, we obtain that $\zeta_{G_2(5)}(3b, 5a, 31a) = \zeta_{G_2(5)}(3b, 5a, 31b) = 31 = \zeta_{G_2(5)}(3b, 5a, 31c) = \zeta_{G_2(5)}(3b, 5a, 31d) = \zeta_{G_2(5)}(3b, 5a, 31e)$ and $\zeta_{G_2(5)}(3b, 5b, 31a) = \zeta_{G_2(5)}(3b, 5b, 31b) = 465 = \zeta_{G_2(5)}(3b, 5b, 31c) = \zeta_{G_2(5)}(3b, 5b, 31d) = \zeta_{G_2(5)}(3b, 5b, 31e)$.

From Table 9, we have that $\zeta_{5^3.L_3(5)}(3a, 5b, 31a) = \zeta_{5^3.L_3(5)}(3a, 5b, 31b) = \zeta_{5^3.L_3(5)}(3a, 5b, 31c) = \zeta_{5^3.L_3(5)}(3a, 5b, 31d) = \zeta_{5^3.L_3(5)}(3a, 5b, 31e) = 155 = \zeta_{5^3.L_3(5)}(3a, 5b, 31f) = \zeta_{5^3.L_3(5)}(3a, 5b, 31g) = \zeta_{5^3.L_3(5)}(3a, 5b, 31h) = \zeta_{5^3.L_3(5)}(3a, 5b, 31i) = \zeta_{5^3.L_3(5)}(3a, 5b, 31j)$. We thus obtain that $\zeta^*_{Ly}(3B, 5A, 31A) = 124651 - 31 - 465 - 155 - 155 = 123845$ proving that Ly is (3B,5A,31A)-generated, $\zeta^*_{Ly}(3B, 5A, 31B) = 124651 - 31 - 465 - 155 = 124000$ proving that Ly is $(3B, 5A, 31B)$-generated, $\zeta^*_{Ly}(3B, 5A, 31C) = 124651 - 31 - 465 - 155 - 155 - 155 = 123690$ proving that Ly is $(3B, 5A, 31C)$-generated, $\zeta^*_{Ly}(3B, 5A, 31D) = 124651 - 31 - 465 = 124155$ proving that Ly is $(3B, 5A, 31D)$-generated and $\zeta^*_{Ly}(3B, 5A, 31E) = 124651 - 31 - 465 - 155 - 155 - 155 - 155 = 123535$ proving that Ly is $(3B, 5A, 31E)$-generated.

By Table 28, we have that $\zeta_{Ly}(3B, 5B, 31A) = \zeta_{Ly}(3B, 5B, 31B) = 78192850 = \zeta_{Ly}(3B, 5B, 31C) = \zeta_{Ly}(3B, 5B, 31D) = \zeta_{Ly}(3B, 5B, 31E)$. By Table 7, we obtain that $\zeta_{G_2(5)}(3b, 5c, 31a) = \zeta_{G_2(5)}(3b, 5c, 31b) = 2170 = \zeta_{G_2(5)}(3b, 5c, 31c) = \zeta_{G_2(5)}(3b, 5c, 31d) = \zeta_{G_2(5)}(3b, 5c, 31e)$, $\zeta_{G_2(5)}(3b, 5d, 31a) = \zeta_{G_2(5)}(3b, 5d, 31b) = 4030 = \zeta_{G_2(5)}(3b, 5d, 31c) = \zeta_{G_2(5)}(3b, 5d, 31d) = \zeta_{G_2(5)}(3b, 5d, 31e)$ and $\zeta_{G_2(5)}(3b, 5e, 31a) = \zeta_{G_2(5)}(3b, 5e, 31b) = 6975 = \zeta_{G_2(5)}(3b, 5e, 31c) = \zeta_{G_2(5)}(3b, 5e, 31d) = \zeta_{G_2(5)}(3b, 5e, 31e)$.

By Table 9, we obtain that $\zeta_{5^3.L_3(5)}(3a, 5c, 31a) = \zeta_{5^3.L_3(5)}(3a, 5c, 31b) = \zeta_{5^3.L_3(5)}(3a, 5c, 31c) = \zeta_{5^3.L_3(5)}(3a, 5c, 31d) = \zeta_{5^3.L_3(5)}(3a, 5c, 31e) = \zeta_{5^3.L_3(5)}(3a, 5c, 31f) = \zeta_{5^3.L_3(5)}(3a, 5c, 31g) = \zeta_{5^3.L_3(5)}(3a, 5c, 31h) = \zeta_{5^3.L_3(5)}(3a, 5c, 31i) = \zeta_{5^3.L_3(5)}(3a, 5c, 31j) = 310 = \zeta_{5^3.L_3(5)}(3a, 5d, 31a) = \zeta_{5^3.L_3(5)}(3a, 5d, 31b) = \zeta_{5^3.L_3(5)}(3a, 5d, 31c) = \zeta_{5^3.L_3(5)}(3a, 5d, 31d) = \zeta_{5^3.L_3(5)}(3a, 5d, 31e) = \zeta_{5^3.L_3(5)}(3a, 5d, 31f) = \zeta_{5^3.L_3(5)}(3a, 5d, 31g) = \zeta_{5^3.L_3(5)}(3a, 5d, 31h) = \zeta_{5^3.L_3(5)}(3a, 5d, 31i) = \zeta_{5^3.L_3(5)}(3a, 5d, 31j)$.

Thus, we obtain that $\zeta^*_{Ly}(3B, 5B, 31A) = 78192850 - 2170 - 4030 - 6975 - 310 - 310 - 310 - 310 = 78178435$ proving that Ly is $(3B, 5B, 31A)$-generated, $\zeta^*_{Ly}(3B, 5B, 31B) = 78192850 - 2170 - 4030 - 6975 -$

$310 - 310 = 78179055$ proving that Ly is $(3B, 5B, 31B)$-generated, $\zeta^*_{Ly}(3B, 5B, 31C) = 78192850 - 2170 - 4030 - 6975 - 310 - 310 - 310 - 310 - 310 - 310 = 78177815$ proving that Ly is $(3B, 5B, 31C)$-generated, $\zeta^*_{Ly}(3B, 5B, 31D) = 78192850 - 2170 - 4030 - 6975 = 78179675$ proving that Ly is $(3B, 5B, 31D)$-generated and $\zeta^*_{Ly}(3B, 5B, 31E) = 78192850 - 2170 - 4030 - 6975 - 310 - 310 - 310 - 310 - 310 - 310 - 310 - 310 = 78177195$ proving that Ly is $(3B, 5B, 31E)$-generated. Hence the proof is complete.

Proposition 4.4. Ly is
(i) not $(3A, 5A, 37)$-generated
(ii) $(3A, 5B, 37), (3B, 5A, 37), (3B, 5B, 37)$-generated

Proof (i) By Table 28, we have that $\zeta_{Ly}(3A, 5A, 37A) = 0 = \zeta_{Ly}(3A, 5A, 37B)$, proving (i).

(ii) By Table 28, we have that $\zeta_{Ly}(3A, 5B, 37A) = 8658 = \zeta_{Ly}(3A, 5B, 37B), \zeta_{Ly}(3B, 5A, 37A) = 141192 = \zeta_{Ly}(3B, 5A, 37B)$ and $\zeta_{Ly}(3B, 5B, 37A) = 78319010 = \zeta_{Ly}(3B, 5B, 37B)$. There is no contribution from any maximal subgroup because none contains elements of order 37. Thus we obtain that $\zeta^*_{Ly}(3A, 5B, 37A) = 8658 = \zeta^*_{Ly}(3A, 5B, 37B)$, $\zeta^*_{Ly}(3B, 5A, 37A) = 141192 = \zeta^*_{Ly}(3B, 5A, 37B)$ and $\zeta^*_{Ly}(3B, 5B, 37A) = 78319010 = \zeta^*_{Ly}(3B, 5B, 37B)$, proving (ii).

Proposition 4.5. Ly is
(i) not $(3A, 5A, 67)$-generated
(ii) $(3A, 5B, 67), (3B, 5A, 67), (3B, 5B, 67)$-generated

Proof (i) By Table 28, we have that $\zeta_{Ly}(3A, 5A, 67A) = 0 = \zeta_{Ly}(3A, 5A, 67B) = \zeta_{Ly}(3A, 5A, 67C)$, proving (i).

(ii) By Table 28, we have that $\zeta_{Ly}(3A, 5B, 67A) = 7169 = \zeta_{Ly}(3A, 5B, 67B) = \zeta_{Ly}(3A, 5B, 67C)$, $\zeta_{Ly}(3B, 5A, 67A) = 140231 = \zeta_{Ly}(3B, 5A, 67B) = \zeta_{Ly}(3B, 5A, 67C)$ and $\zeta_{Ly}(3B, 5B, 67A) = 78319010 = \zeta_{Ly}(3B, 5B, 67B) = \zeta_{Ly}(3B, 5B, 67C)$. There is no contribution from any maximal subgroup because none contains elements of order 67. Thus we obtain that $\zeta^*_{Ly}(3A, 5B, 67A) = 7169 = \zeta^*_{Ly}(3A, 5B, 67B) = \zeta^*_{Ly}(3A, 5B, 67C), \zeta^*_{Ly}(3B, 5A, 67A) = 140231 = \zeta^*_{Ly}(3B, 5A, 67B) = \zeta^*_{Ly}(3B, 5A, 67C)$ and $\zeta^*_{Ly}(3B, 5B, 67A) = 78319010 = \zeta^*_{Ly}(3B, 5B, 67B) = \zeta^*_{Ly}(3B, 5B, 67C)$ hence (ii) follows and the proof is complete.

Table 28. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3A, 5A, tX)$	$\zeta_{Ly}(3A, 5B, tX)$	$\zeta_{Ly}(3B, 5A, tX)$	$\zeta_{Ly}(3B, 5B, tX)$
$7A$	168	7	10528	148197	76620040
$11A$	66	22	5071	131824	80152600
$11B$	66	22	5071	131824	80152600
$31A$	31	0	4030	124651	78192850
$31B$	31	0	4030	124651	78192850
$31C$	31	0	4030	124651	78192850
$31D$	31	0	4030	124651	78192850
$31E$	31	0	4030	124651	78192850
$37A$	37	0	8658	141192	78319010
$37B$	37	0	8658	141192	78319010
$67A$	67	0	7169	140231	78319010
$67B$	67	0	7169	140231	78319010
$67C$	67	0	7169	140231	78319010

Table 29. *Partial structure constants of $G_2(5)$.*

tX	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(3a, 5a, tX)$	$\zeta_{G_2(5)}(3a, 5b, tX)$	$\zeta_{G_2(5)}(3a, 5c, tX)$
$7a$	21	0	0	21
$31a$	31	0	0	0
$31b$	31	0	0	0
$31c$	31	0	0	0
$31d$	31	0	0	0
$31e$	31	0	0	0

tX	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(3a, 5d, tX)$	$\zeta_{G_2(5)}(3a, 5e, tX)$
$7a$	21	0	11
$31a$	31	0	31
$31b$	31	0	31
$31c$	31	0	31
$31d$	31	0	31
$31e$	31	0	31

$\mathbf{tX}$	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(3b,5a,tX)$	$\zeta_{G_2(5)}(3b,5b,tX)$	$\zeta_{G_2(5)}(3b,5c,tX)$
$7a$	21	21	525	1925
31a	31	31	465	2170
$31b$	31	31	465	2170
$31c$	31	31	465	2170
$31d$	31	31	465	2170
$31e$	31	31	465	2170
$\mathbf{tX}$	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(3b,5d,tX)$	$\zeta_{G_2(5)}(3b,5e,tX)$	
$7a$	21	4900	6300	
$31a$	31	4030	6975	
$31b$	31	4030	6975	
$31c$	31	4030	6975	
$31d$	31	4030	6975	
$31e$	31	4030	6975	

Table 30. *Partial structure constants of* $3{\cdot}McL{:}2$.

$\mathbf{tX}$	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(3a,5a,tX)$	$\zeta_{3\cdot McL:2}(3a,5b,tX)$
$7a$	42	0	0
$11a$	66	0	0
$11b$	66	0	0
$\mathbf{tX}$	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(3b,5a,tX)$	$\zeta_{3\cdot McL:2}(3b,5b,tX)$
$7a$	42	7	280
$11a$	66	22	418
$11b$	66	22	418
$\mathbf{tX}$	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(3c,5a,tX)$	$\zeta_{3\cdot McL:2}(3c,5b,tX)$
$7a$	42	56	364
$11a$	66	22	682
$11b$	66	22	682
$\mathbf{tX}$	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3McL:2}(3d,5a,tX)$	$\zeta_{3McL:2}(3d,5b,tX)$
$7a$	42	595	50400
$11a$	66	1122	34485
$11b$	66	1122	34485

Table 31. *Partial structure constants of* $5^3{\cdot}L_3(5)$.

$\mathbf{tX}$	$\lvert C_{5^3\cdot L_3(5)}(tX)\rvert$	$\zeta_{5^3\cdot L_3(5)}(3a,5a,tX)$	$\zeta_{5^3\cdot L_3(5)}(3a,5b,tX)$
$31a$	31	0	155
$31b$	31	0	155
$31c$	31	0	155
$31d$	31	0	155
$31e$	31	0	155
$31f$	31	0	155
$31g$	31	0	155
$31h$	31	0	155
$31i$	31	0	155
$31j$	31	0	155
$\mathbf{tX}$	$\lvert C_{5^3\cdot L_3(5)}(tX)\rvert$	$\zeta_{5^3\cdot L_3(5)}(3a,5c,tX)$	$\zeta_{5^3\cdot L_3(5)}(3a,5d,tX)$
$31a$	31	310	310
$31b$	31	310	310
$31c$	31	310	310

tX	$\|C_{5^3 \cdot L_3(5)}(tX)\|$	$\zeta_{5^3 \cdot L_3(5)}(3a,5c,tX)$	$\zeta_{5^3 \cdot L_3(5)}(3a,5d,tX)$
$31d$	31	310	310
$31e$	31	310	310
$31f$	31	310	310
$31g$	31	310	310
$31h$	31	310	310
$31i$	31	310	310
$31j$	31	310	310

Table 32. *Partial structure constants of* $2{\cdot}A_{11}$.

tX	$\|C_{2\cdot A_{11}}(tX)\|$	$\zeta_{2\cdot A_{11}}(3a,5a,tX)$	$\zeta_{2\cdot A_{11}}(3a,5b,tX)$
$7a$	168	7	0
$11a$	22	0	11
$11b$	22	0	11
tX	$\|C_{2\cdot A_{11}}(tX)\|$	$\zeta_{2\cdot A_{11}}(3b,5a,tX)$	$\zeta_{2\cdot A_{11}}(3b,5b,tX)$
$7a$	168	21	504
$11a$	22	0	242
$11b$	22	0	242
tX	$\|C_{2\cdot A_{11}}(tX)\|$	$\zeta_{2\cdot A_{11}}(3c,5a,tX)$	$\zeta_{2\cdot A_{11}}(3c,5b,tX)$
$7a$	168	168	1680
$11a$	22	0	2816
$11b$	22	0	2816

Table 33. *Partial structure constants of* $3^5{:}(2\times M_{11})$.

tX	$\|C_{3^5:(2\times M_{11})}(tX)\|$	$\zeta_{3^5:(2\times M_{11})}(3a,5a,tX)$	$\zeta_{3^5:(2\times M_{11})}(3b,5a,tX)$
$11a$	22	0	0
$11b$	22	0	0
tX	$\|C_{3^5:(2\times M_{11})}(tX)\|$	$\zeta_{3^5:(2\times M_{11})}(3c,5a,tX)$	$\zeta_{3^5:(2\times M_{11})}(3d,5a,tX)$
$11a$	22	891	1782
$11b$	22	891	1782

4.2. The $(3,7,r)$-Generations of Ly

We consider all $r \in \{11,31,37,67\}$. The maximal subgroups having any contributions are $G_2(5); 3{\cdot}McL{:}2; 2{\cdot}A_{11}$ because $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ contain elements of order 11, while $G_2(5)$ contains elements of order 31.

Proposition 4.6. Ly is $(3,7,11)$-generated

Proof The maximal subgroups containing elements of order 11 are $3{\cdot}McL{:}2; 2{\cdot}A_{11}$. The classes $11a, 11b$ of $3{\cdot}McL{:}2$ are each contained in one conjugate of $3{\cdot}McL{:}2, 11a, 11b$ of $2{\cdot}A_{11}$ are each contained in three conjugates of $2{\cdot}A_{11}$. By Table 34, we have that $\zeta_{Ly}(3A,7A,11A) = 104940 = \zeta_{Ly}(3A,7A,11B)$. Only $3{\cdot}McL{:}2$ meets the $3A,7A,11A,11B$ classes of Ly and by Table 8, we have that $\zeta_{3\cdot McL:2}(3b,7a,11a) = 1452 = \zeta_{3\cdot McL:2}(3b,7a,11b)$. We thus obtain that $\zeta^*_{Ly}(3A,7A,11A) = 104940 - 1452 = 103488 = \zeta^*_{Ly}(3A,7A,11B)$, proving that Ly is $(3A,7A,11)$-generated.

By Table 34, we have that $\zeta_{Ly}(3B,7A,11A) = 1767039153 = \zeta_{Ly}(3B,7A,11B)$. By Table 8, we have that $\zeta_{3\cdot McL:2}(3c,7a,11a) = 2904 = \zeta_{3\cdot McL:2}(3c,7a,11b)$ and $\zeta_{3\cdot McL:2}(3d,7a,11a) = 132264 = \zeta_{3\cdot McL:2}(3d,7a,11b)$. By Table 10, we have that $\zeta_{2\cdot A_{11}}(3c,7a,11a) = 693 = \zeta_{2\cdot A_{11}}(3c,7a,11b)$. Thus we obtain that $\zeta^*_{Ly}(3B,7A,11A) = 1767039153 - 1(2904) - 1(132264) - 3(693) = 1766901906 = \zeta^*_{Ly}(3B,7A,11B)$, proving that Ly is $(3A,7A,11)$-generated.

Proposition 4.7. Ly is $(3,7,31)$-generated.

Proof By Table 34, we have that $\zeta_{Ly}(3A,7A,31A) = \zeta_{Ly}(3A,7A,31B) = 114390 = \zeta_{Ly}(3A,7A,31C) = \zeta_{Ly}(3A,7A,31D) = \zeta_{Ly}(3A,7A,31E)$. The only maximal subgroup containing elements of order 31 is $G_2(5)$. Thus $31a,31b,31c,31d,31e$ of $G_2(5)$ are each contained in one conjugate of $G_2(5)$. By Table 7, we have that $\zeta_{G_2(5)}(3a,7a,31a) = \zeta_{G_2(5)}(3a,7a,31b) = 806 = \zeta_{G_2(5)}(3a,7a,31c) = \zeta_{G_2(5)}(3a,7a,31d) = \zeta_{G_2(5)}(3a,7a,31e)$. Thus we obtain that $\zeta^*_{Ly}(3A,7A,31A) = 114390 - 806 = 113584 = \zeta^*_{Ly}(3A,7A,31B) = \zeta^*_{Ly}(3A,7A,31C) =$

$\zeta^*_{Ly}(3A,7A,31D) = \zeta^*_{Ly}(3A,7A,31E)$, proving that Ly is $(3A,7A,31)$-generated.

By Table 34, we have that $\zeta_{Ly}(3B,7A,31A) = \zeta_{Ly}(3B,7A,31B) = 1761121098 = \zeta_{Ly}(3B,7A,31C) = \zeta_{Ly}(3B,7A,31D) = \zeta_{Ly}(3A,7B,31E)$. By Table 7, we have that $\zeta_{G_2(5)}(3b,7a,31a) = \zeta_{G_2(5)}(3b,7a,31b) = 387376 = \zeta_{G_2(5)}(3b,7a,31c) = \zeta_{G_2(5)}(3b,7a,31d) = \zeta_{G_2(5)}(3b,7a,31e)$. We thus obtain that $\zeta^*_{Ly}(3B,7A,31A) = 1761121098 - 387376 = 1760733722 = \zeta^*_{Ly}(3B,7A,31B) = \zeta^*_{Ly}(3B,7A,31C) = \zeta^*_{Ly}(3B,7A,31D) = \zeta^*_{Ly}(3B,7A,31E)$, proving that Ly is $(3B,7A,31)$-generated.

Proposition 4.8. Ly is $(3,7,37)$-generated.

Proof By Table 34, we have that $\zeta_{Ly}(3A,7A,37A) = 135050 = \zeta_{Ly}(3A,7A,37B)$, and $\zeta_{Ly}(3B,7A,37A) = 1750265353 = \zeta_{Ly}(3B,7A,37B)$. There is no contribution from any maximal subgroup because none contains elements of order 37. Thus we obtain that $\zeta^*_{Ly}(3A,7A,37A) = 135050 = \zeta^*_{Ly}(3A,7A,37B)$ and $\zeta^*_{Ly}(3B,7A,37A) = 1750265353 = \zeta^*_{Ly}(3B,7A,37B)$ proving that Ly is $(3,7,37)$-generated.

Proposition 4.9. Ly is $(3,7,67)$-generated.

Proof From Table 34, we have that $\zeta_{Ly}(3A,7A,67A) = 133330 = \zeta_{Ly}(3A,7A,67B) = \zeta_{Ly}(3A,7A,67C)$ and $\zeta_{Ly}(3B,7A,67A) = 1749468088 = \zeta_{Ly}(3B,7A,67B) = \zeta_{Ly}(3B,7A,67C)$. There is no contribution from any maximal subgroup because none contains elements of order 67. Thus we obtain that $\zeta^*_{Ly}(3A,7A,67A) = 133330 = \zeta^*_{Ly}(3A,7A,67B) = \zeta^*_{Ly}(3A,7A,67C)$ and $\zeta^*_{Ly}(3B,7A,67A) = 1749468088 = \zeta^*_{Ly}(3B,7A,67B) = \zeta^*_{Ly}(3B,7A,67C)$ proving that Ly is $(3,7,67)$-generated.

Table 34. *Partial structure constants of* Ly.

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3A,7A,tX)$	$\zeta_{Ly}(3B,7A,tX)$
$11A$	66	104940	1767039153
$11B$	66	104940	1767039153
$31A$	31	114390	1761121098
$31B$	31	114390	1761121098
$31C$	31	114390	1761121098
$31D$	31	114390	1761121098
$31E$	31	114390	1761121098
$37A$	37	135050	1750265353
$37B$	37	135050	1750265353
$67A$	67	133330	1749468088
$67B$	67	133330	1749468088
$67C$	67	133330	1749468088

Table 35. *Partial structure constants of* $G_2(5)$.

tX	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(3a,7a,tX)$	$\zeta_{G_2(5)}(3b,7a,tX)$
$31a$	31	806	387376
$31b$	31	806	387376
$31c$	31	806	387376
$31d$	31	806	387376
$31e$	31	806	387376

Table 36. *Partial structure constants of* $3{\cdot}McL{:}2$.

tX	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(3a,7a,tX)$	$\zeta_{3\cdot McL:2}(3b,7a,tX)$
$11a$	66	0	1452
$11b$	66	0	1452

tX	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(3c,7a,tX)$	$\zeta_{3\cdot McL:2}(3d,7a,tX)$
$11a$	66	2904	132264
$11b$	66	2904	132264

Table 37. *Partial structure constants of* $2{\cdot}A_{11}$.

tX	$\lvert C_{2\cdot A_{11}}(tX)\rvert$	$\zeta_{2\cdot A_{11}}(3a,7a,tX)$	$\zeta_{2\cdot A_{11}}(3b,7a,tX)$	$\zeta_{2\cdot A_{11}}(3c,7a,tX)$
11a	22	0	0	693
11b	22	0	0	693

4.3. The $(3,11,r)$-Generations of Ly

We shall consider all $r \in \{31,37,67\}$. No maximal subgroup has any contribution here since there is none containing elements of orders 31, 37 and 67.

Proposition 4.10. Ly is $(3,11,31)$-generated.

Proof By Table 38, we have that $\zeta_{Ly}(3A,11A,31A) = \zeta_{Ly}(3A,11A,31B) = \zeta_{Ly}(3A,11A,31C) = \zeta_{Ly}(3A,11A,31D) = \zeta_{Ly}(3A,11A,31E) = 294810 = \zeta_{Ly}(3A,11B,31A) = \zeta_{Ly}(3A,11B,31B) = \zeta_{Ly}(3A,11B,31C) = \zeta_{Ly}(3A,11B,31D) = \zeta_{Ly}(3A,11B,31E)$ and $\zeta_{Ly}(3B,11A,31A) = \zeta_{Ly}(3B,11A,31B) = \zeta_{Ly}(3B,11A,31C) = \zeta_{Ly}(3B,11A,31D) = \zeta_{Ly}(3B,11A,31E) = 4485289467 = \zeta_{Ly}(3B,11B,31A) = \zeta_{Ly}(3B,11B,31B) = \zeta_{Ly}(3B,11B,31C) = \zeta_{Ly}(3B,11B,31D) = \zeta_{Ly}(3B,11B,31E)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(3A,11A,31A) = \zeta^*_{Ly}(3A,11A,31B) = \zeta^*_{Ly}(3A,11A,31C) = \zeta^*_{Ly}(3A,11A,31D) = \zeta^*_{Ly}(3A,11A,31E) = 294810 = \zeta^*_{Ly}(3A,11B,31A) = \zeta^*_{Ly}(3A,11B,31B) = \zeta^*_{Ly}(3A,11B,31C) = \zeta^*_{Ly}(3A,11B,31D) = \zeta^*_{Ly}(3A,11B,31E)$ and $\zeta^*_{Ly}(3B,11A,31A) = \zeta^*_{Ly}(3B,11A,31B) = \zeta^*_{Ly}(3B,11A,31C) = \zeta^*_{Ly}(3B,11A,31D) = \zeta^*_{Ly}(3B,11A,31E) = 4485289467 = \zeta^*_{Ly}(3B,11B,31A) = \zeta^*_{Ly}(3B,11B,31B) = \zeta^*_{Ly}(3B,11B,31C) = \zeta^*_{Ly}(3B,11B,31D) = \zeta^*_{Ly}(3B,11B,31E)$ proving that Ly is $(3,11,31)$-generated.

Proposition 4.11. Ly is $(3,11,37)$-generated.

Proof By Table 38 , we have that $\zeta_{Ly}(3A,11A,37A) = \zeta_{Ly}(3A,11A,37B) = 274096 = \zeta_{Ly}(3A,11B,37A) = \zeta_{Ly}(3A,11B,37B)$ and $\zeta_{Ly}(3B,11A,37A) = \zeta_{Ly}(3B,11A,37B) = 4487698476 = \zeta_{Ly}(3B,11B,37A) = \zeta_{Ly}(3B,11B,37B)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(3A,11A,37A) = \zeta^*_{Ly}(3A,11A,37B) = 274096 = \zeta^*_{Ly}(3A,11B,37A) = \zeta^*_{Ly}(3A,11B,37B)$ and $\zeta^*_{Ly}(3B,11A,37A) = \zeta^*_{Ly}(3B,11A,37B) = 4487698476 = \zeta^*_{Ly}(3B,11B,37A) = \zeta^*_{Ly}(3B,11B,37B)$ proving that Ly is $(3,11,37)$-generated.

Proposition 4.12. Ly is $(3,11,67)$-generated.

Proof By Table 38, we have that $\zeta_{Ly}(3A,11A,67A) = \zeta_{Ly}(3A,11A,67B) = \zeta_{Ly}(3A,11A,67C) = 276777 = \zeta_{Ly}(3A,11B,67A) = \zeta_{Ly}(3A,11B,67B) = \zeta_{Ly}(3A,11B,67C)$ and $\zeta_{Ly}(3B,11A,67A) = \zeta_{Ly}(3B,11A,67B) = \zeta_{Ly}(3B,11A,67C) = 4489529166 = \zeta_{Ly}(3B,11B,67A) = \zeta_{Ly}(3B,11B,67B) = \zeta_{Ly}(3B,11B,67C)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(3A,11A,67A) = \zeta^*_{Ly}(3A,11A,67B) = \zeta^*_{Ly}(3A,11A,67C) = 276777 = \zeta^*_{Ly}(3A,11B,67A) = \zeta^*_{Ly}(3A,11B,67B) = \zeta^*_{Ly}(3A,11B,67C)$ and $\zeta^*_{Ly}(3B,11A,67A) = \zeta^*_{Ly}(3B,11A,67B) = \zeta^*_{Ly}(3B,11A,67C) = 4489529166 = \zeta^*_{Ly}(3B,11B,67A) = \zeta^*_{Ly}(3B,11B,67B) = \zeta^*_{Ly}(3B,11B,67C)$ proving that Ly is $(3,11,67)$-generated.

Table 38. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3A,11A,tX)$	$\zeta_{Ly}(3A,11B,tX)$	$\zeta_{Ly}(3B,11A,tX)$	$\zeta_{Ly}(3B,11B,tX)$
$31A$	31	294810	294810	4485289467	4485289467
$31B$	31	294810	294810	4485289467	4485289467
$31C$	31	294810	294810	4485289467	4485289467
$31D$	31	294810	294810	4485289467	4485289467
$31E$	31	294810	294810	4485289467	4485289467
$37A$	37	274096	274096	4487698476	4487698476
$37B$	37	274096	274096	4487698476	4487698476
$67A$	67	276777	276777	4489529166	4489529166
$67B$	67	276777	276777	4489529166	4489529166
$67C$	67	276777	276777	4489529166	4489529166

4.4. The $(3,31,r)$-Generations of Ly

We shall consider all $r \in \{37,67\}$. No maximal subgroup has any contribution here since none contains elements of orders 37 and 67.

Proposition 4.13. Ly is $(3,31,37)$-generated.

Proof By Table 39, we have that $\zeta_{Ly}(3A,31A,37A) = \zeta_{Ly}(3A,31A,37B) = \zeta_{Ly}(3A,31B,37A) = \zeta_{Ly}(3A,31B,37B) = \zeta_{Ly}(3A,31C,37A) = 629000 = \zeta_{Ly}(3A,31C,37B) = \zeta_{Ly}(3A,31D,37A) = \zeta_{Ly}(3A,31D,37B) = \zeta_{Ly}(3A,31E,37A) = \zeta_{Ly}(3A,31E,37B)$ and $\zeta_{Ly}(3B,31A,37A) = \zeta_{Ly}(3B,31A,37B) = \zeta_{Ly}(3B,31B,37A) = \zeta_{Ly}(3B,31B,37B) = \zeta_{Ly}(3B,31C,37A) = 9549237500 = \zeta_{Ly}(3B,31C,37B) = \zeta_{Ly}(3B,31D,37A) = \zeta_{Ly}(3B,31D,37B) = \zeta_{Ly}(3B,31E,37A) = \zeta_{Ly}(3B,31E,37B)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(3A,31A,37A) = \zeta^*_{Ly}(3A,31A,37B) = \zeta^*_{Ly}(3A,31B,37A) = \zeta^*_{Ly}(3A,31B,37B) = \zeta^*_{Ly}(3A,31C,37A) = 629000 = \zeta^*_{Ly}(3A,31C,37B) = \zeta^*_{Ly}(3A,31D,37A) = \zeta^*_{Ly}(3A,31D,37B) = \zeta^*_{Ly}(3A,31E,37A) = \zeta^*_{Ly}(3A,31E,37B)$ and $\zeta^*_{Ly}(3B,31A,37A) = \zeta^*_{Ly}(3B,31A,37B) = \zeta^*_{Ly}(3B,31B,37A) = \zeta^*_{Ly}(3B,31B,37B) = \zeta^*_{Ly}(3B,31C,37A) = 9549237500 = \zeta^*_{Ly}(3B,31C,37B) =$

$\zeta^*_{Ly}(3B,31D,37A) = \zeta^*_{Ly}(3B,31D,37B) = \zeta^*_{Ly}(3B,31E,37A) = \zeta^*_{Ly}(3B,31E,37B)$ proving that Ly is $(3,31,37)$ generated.

Proposition 4.14. Ly is $(3,31,67)$-generated.

Proof By Table 39, we get that $\zeta_{Ly}(3A,31A,67A) = \zeta_{Ly}(3A,31A,67B) = \zeta_{Ly}(3A,31A,67C) = \zeta_{Ly}(3A,31B,67A) = \zeta_{Ly}(3A,31B,67B) = \zeta_{Ly}(3A,31B,67C) = \zeta_{Ly}(3A,31C,67A) = 619750 = \zeta_{Ly}(3A,31C,67B) = \zeta_{Ly}(3A,31C,67C) = \zeta_{Ly}(3A,31D,67A) = \zeta_{Ly}(3A,31D,67B) = \zeta_{Ly}(3A,31D,67C) = \zeta_{Ly}(3A,31E,67A) = \zeta_{Ly}(3A,31E,67B) = \zeta_{Ly}(3A,31E,67C)$ and $\zeta_{Ly}(3B,31A,67A) = \zeta_{Ly}(3B,31A,67B) = \zeta_{Ly}(3B,31A,67C) = \zeta_{Ly}(3B,31B,67A) = \zeta_{Ly}(3B,31B,67B) = \zeta_{Ly}(3B,31B,67C) = \zeta_{Ly}(3B,31C,67A) = 9544150000 = \zeta_{Ly}(3B,31C,67B) = \zeta_{Ly}(3B,31C,67C) = \zeta_{Ly}(3B,31D,67A) = \zeta_{Ly}(3B,31D,67B) = \zeta_{Ly}(3B,31D,67C) = \zeta_{Ly}(3B,31E,67A) = \zeta_{Ly}(3B,31E,67B) = \zeta_{Ly}(3B,31E,67C)$. Since there is no contribution from any of the maximal subgroups, it follows that $\zeta^*_{Ly}(3A,31A,67A) = \zeta^*_{Ly}(3A,31A,67B) = \zeta^*_{Ly}(3A,31A,67C) = \zeta^*_{Ly}(3A,31B,67A) = \zeta^*_{Ly}(3A,31B,67B) = \zeta^*_{Ly}(3A,31B,67C) = \zeta^*_{Ly}(3A,31C,67A) = 619750 = \zeta^*_{Ly}(3A,31C,67B) = \zeta^*_{Ly}(3A,31C,67C) = \zeta^*_{Ly}(3A,31D,67A) = \zeta^*_{Ly}(3A,31D,67B) = \zeta^*_{Ly}(3A,31D,67C) = \zeta^*_{Ly}(3A,31E,67A) = \zeta^*_{Ly}(3A,31E,67B) = \zeta^*_{Ly}(3A,31E,67C)$ and $\zeta^*_{Ly}(3B,31A,67A) = \zeta^*_{Ly}(3B,31A,67B) = \zeta^*_{Ly}(3B,31A,67C) = \zeta^*_{Ly}(3B,31B,67A) = \zeta^*_{Ly}(3B,31B,67B) = \zeta^*_{Ly}(3B,31B,67C) = \zeta^*_{Ly}(3B,31C,67A) = 9544150000 = \zeta^*_{Ly}(3B,31C,67B) = \zeta^*_{Ly}(3B,31C,67C) = \zeta^*_{Ly}(3B,31D,67A) = \zeta^*_{Ly}(3B,31D,67B) = \zeta^*_{Ly}(3B,31D,67C) = \zeta^*_{Ly}(3B,31E,67A) = \zeta^*_{Ly}(3B,31E,67B) = \zeta^*_{Ly}(3B,31E,67C)$ proving that Ly is $(3,31,67)$-generated.

Table 39. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3A,31A,tX)$	$\zeta_{Ly}(3A,31B,tX)$	$\zeta_{Ly}(3A,31C,tX)$
$37A$	37	629000	629000	629000
$37B$	37	629000	629000	629000
$67A$	67	619750	619750	619750
$67B$	67	619750	619750	619750
$67C$	67	619750	619750	619750

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3A,31D,tX)$	$\zeta_{Ly}(3A,31E,tX)$
$37A$	37	629000	629000
$37B$	37	629000	629000
$67A$	67	619750	619750
67B	67	619750	619750
$67C$	67	619750	619750

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3B,31A,tX)$	$\zeta_{Ly}(3B,31B,tX)$	$\zeta_{Ly}(3B,31C,tX)$
$37A$	37	9549237500	9549237500	9549237500
$37B$	37	9549237500	9549237500	9549237500
$67A$	67	9544150000	9544150000	9544150000
$67B$	67	9544150000	9544150000	9544150000
$67C$	67	9544150000	9544150000	9544150000

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3B,31D,tX)$	$\zeta_{Ly}(3B,31E,tX)$
$37A$	37	9549237500	9549237500
$37B$	37	9549237500	9549237500
$67A$	67	9544150000	9544150000
$67B$	67	9544150000	9544150000
$67C$	67	9544150000	9544150000

4.5. The $(3,37,r)$-Generations of Ly

We shall only consider $r = 67$ in this case. No maximal subgroup has any contribution here since there is none containing elements of order 67.

Proposition 4.15. Ly is $(3,37,67)$-generated.

Proof By Table 40, we obtain that $\zeta_{Ly}(3A,37A,67A) = \zeta_{Ly}(3A,37A,67B) = \zeta_{Ly}(3A,37A,67C) = 563537 = \zeta_{Ly}(3A,37B,67A) = \zeta_{Ly}(3A,37B,67B) = \zeta_{Ly}(3A,37B,67C), \zeta_{Ly}(3B,37A,67A) = \zeta_{Ly}(3B,37A,67B) = 7968417236 = \zeta_{Ly}(3B,37A,67C)$ and $\zeta_{Ly}(3B,37B,67A) = \zeta_{Ly}(3B,37B,67B) = 7971283460 = \zeta_{Ly}(3B,37B,67C)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(3A,37A,67A) = \zeta^*_{Ly}(3A,37A,67B) = \zeta^*_{Ly}(3A,37A,67C) = 563537 = \zeta^*_{Ly}(3A,37B,67A) = \zeta^*_{Ly}(3A,37B,67B) =$

$\zeta^*_{Ly}(3A,37B,67C), \zeta^*_{Ly}(3B,37A,67A) = \zeta^*_{Ly}(3B,37A,67B) = 7968417236 = \zeta^*_{Ly}(3B,37A,67C)$ and $\zeta^*_{Ly}(3B,37B,67A) = \zeta^*_{Ly}(3B,37B,67B) = 7971283460 = \zeta^*_{Ly}(3B,37B,67C)$ proving that Ly is $(3,37,67)$-generated.

Table 40. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(3A,37A,tX)$	$\zeta_{Ly}(3A,37B,tX)$	$\zeta_{Ly}(3B,37A,tX)$	$\zeta_{Ly}(3B,37B,tX)$
$67A$	67	563537	563537	7968417236	7971283460
$67B$	67	563537	563537	7968417236	7971283460
$67C$	67	563537	563537	7968417236	7971283460

5. The $(5,q,r)$-Generations of Ly

We shall consider all $q \in \{7,11,31,37\}, r \in \{11,31,37,67\}$.

5.1. The $(5,7,r)$-Generations of Ly

We shall consider all $r \in \{11,31,37,67\}$. The maximal subgroups having any contributions are $G_2(5); 3{\cdot}McL{:}2; 2{\cdot}A_{11}$ because $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ contain elements of order 11, while $G_2(5)$ contains elements of order 31.

Proposition 5.1. Ly is $(5,7,11)$-generated.

Proof By Table 41, we have that $\zeta_{Ly}(5A,7A,11A) = 139215681 = \zeta_{Ly}(5A,7A,11B)$. The maximal subgroups containing elements of order 11 are $3{\cdot}McL{:}2; 2{\cdot}A_{11}$ only. However only $3{\cdot}McL{:}2$ has contributions here since $2{\cdot}A_{11}$ has its relevant structure constants all zero. We obtain that $11a, 11b$ of $3{\cdot}McL{:}2$ are each contained in one conjugate of $3{\cdot}McL{:}2$. By Table 8, we obtain that $\zeta_{3\cdot McL:2}(5a,7a,11a) = 57024 = \zeta_{3\cdot McL:2}(5a,7a,11b)$. From this we obtain that $\zeta^*_{Ly}(5A,7A,11A) = 139215681 - 57024 = 139158657 = \zeta^*_{Ly}(5A,7A,11B)$ proving that Ly is $(5A,7A,11)$-generated.

By Table 41, we have $\zeta_{Ly}(5B,7A,11A) = 81836325450 = \zeta_{Ly}(5B,7A,11B)$. We have that $3{\cdot}McL{:}2$ and $2{\cdot}A_{11}$ meet the $5B, 7A, 11A, 11B$ classes of Ly. However $11a, 11b$ of $2{\cdot}A_{11}$ are each contained in three conjugates of $2{\cdot}A_{11}$. By Tables 8 and 10, we obtain that $\zeta_{3\cdot McL:2}(5b,7a,11a) = 1710720 = \zeta_{3\cdot McL:2}(5b,7a,11b)$ and $\zeta_{2\cdot A_{11}}(5b,7a,11a) = 4026 = \zeta_{2\cdot A_{11}}(5b,7a,11b)$. From this we obtain that $\zeta^*_{Ly}(5B,7A,11A) = 81836325450 - 1(1710720) - 3(4026) = 81834602652 = \zeta^*_{Ly}(5B,7A,11B)$, proving that Ly is $(5B,7A,11)$-generated.

Proposition 5.2. Ly is $(5,7,31)$-generated.

Proof Using Table 41, we have that $\zeta_{Ly}(5A,7A,31A) = \zeta_{Ly}(5A,7A,31B) = 136944918 = \zeta_{Ly}(5A,7A,31C) = \zeta_{Ly}(5A,7A,31D) = \zeta_{Ly}(5A,7A,31E)$. The only maximal subgroup containing elements of order 31 is $G_2(5)$. However $31a, 31b, 31c, 31d, 31e$ of $G_2(5)$ are each contained in one conjugate of $G_2(5)$. By Table 7, we have that $\zeta_{G_2(5)}(5a,7a,31a) = \zeta_{G_2(5)}(5a,7a,31b) = 806 = \zeta_{G_2(5)}(5a,7a,31c) = \zeta_{G_2(5)}(5a,7a,31d) = \zeta_{G_2(5)}(5a,7a,31e)$ and $\zeta_{G_2(5)}(5b,7a,31a) = \zeta_{G_2(5)}(5b,7a,31b) = 18600 = \zeta_{G_2(5)}(5b,7a,31c) = \zeta_{G_2(5)}(5b,7a,31d) = \zeta_{G_2(5)}(5b,7a,31e)$, thus rendering $\zeta^*_{Ly}(5A,7A,31A) = 136944918 - 806 - 18600 = 136925512 = \zeta^*_{Ly}(5A,7A,31B) = \zeta^*_{Ly}(5A,7A,31C) = \zeta^*_{Ly}(5A,7A,31D) = \zeta^*_{Ly}(5A,7A,31E)$.

By Table 41, we have that $\zeta_{Ly}(5B,7A,31A) = \zeta_{Ly}(5B,7A,31B) = 82166950800 = \zeta_{Ly}(5B,7A,31C) = \zeta_{Ly}(5B,7A,31D) = \zeta_{Ly}(5B,7A,31E)$. By Table 7, we have that $\zeta_{G_2(5)}(5c,7a,31a) = \zeta_{G_2(5)}(5c,7a,31b) = 75950 = \zeta_{G_2(5)}(5c,7a,31c) = \zeta_{G_2(5)}(5c,7a,31d) = \zeta_{G_2(5)}(5c,7a,31e), \zeta_{G_2(5)}(5d,7a,31a) = \zeta_{G_2(5)}(5d,7a,31b) = 150350 = \zeta_{G_2(5)}(5d,7a,31c) = \zeta_{G_2(5)}(5d,7a,31d) = \zeta_{G_2(5)}(5d,7a,31e)$ and $\zeta_{G_2(5)}(5d,7a,31a) = \zeta_{G_2(5)}(5d,7a,31b) = 218550 = \zeta_{G_2(5)}(5d,7a,31c) = \zeta_{G_2(5)}(5d,7a,31d) = \zeta_{G_2(5)}(5d,7a,31e)$, rendering $\zeta^*_{Ly}(5B,7A,31A) = 82166950800 - 75950 - 150350 - 218550 = 82166505950 = \zeta^*_{Ly}(5B,7A,31B) = \zeta^*_{Ly}(5B,7A,31C) = \zeta^*_{Ly}(5B,7A,31D) = \zeta^*_{Ly}(5B,7A,31E)$. Hence the proof is complete.

Proposition 5.3. Ly is $(5,7,37)$-generated.

Proof By Table 41, we have that $\zeta_{Ly}(5A,7A,37A) = 132934747 = \zeta_{Ly}(5A,7A,37B)$ and $\zeta_{Ly}(5B,7A,37A) = 82809311500 = \zeta_{Ly}(5B,7A,37B)$. There is no contribution from any maximal subgroup because none contains elements of order 37. Thus we obtain that $\zeta^*_{Ly}(5A,7A,37A) = 132934747 = \zeta^*_{Ly}(5A,7A,37B)$ and $\zeta^*_{Ly}(5B,7A,37A) = 82809311500 = \zeta^*_{Ly}(5B,7A,37B)$ proving that Ly is $(5,7,37)$-generated.

Proposition 5.4. Ly is $(5,7,67)$-generated.

Proof By Table 41, we have that $\zeta_{Ly}(5A,7A,67A) = 132495314 = \zeta_{Ly}(5A,7A,67B) = \zeta_{Ly}(5A,7A,67C)$ and $\zeta_{Ly}(5B,7A,67A) = 82829855500 = \zeta_{Ly}(5B,7A,67B) = \zeta_{Ly}(5B,7A,67C)$. There is no contribution from any maximal subgroup because none contains elements of order 67. Thus we obtain that $\zeta^*_{Ly}(5A,7A,67A) = 132495314 = \zeta^*_{Ly}(5A,7A,67B) = \zeta^*_{Ly}(5A,7A,67C)$ and $\zeta^*_{Ly}(5B,7A,67A) = 82829855500 = \zeta^*_{Ly}(5B,7A,67B) = \zeta^*_{Ly}(5B,7A,67C)$ proving that Ly is $(5,7,67)$-generated.

Table 41. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(5A,7A,tX)$	$\zeta_{Ly}(5B,7A,tX)$
11A	66	139215681	81836325450
11B	66	139215681	81836325450

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(5A,7A,tX)$	$\zeta_{Ly}(5B,7A,tX)$
31A	31	136944918	82166950800
31B	31	136944918	82166950800
31C	31	136944918	82166950800
31D	31	136944918	82166950800
31E	31	136944918	82166950800
37A	37	132934747	82809311500
37B	37	132934747	82809311500
67A	67	132495314	82829855500
67B	67	132495314	82829855500
67C	67	132495314	82829855500

Table 42. *Partial structure constants of* $G_2(5)$.

tX	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(5a,7a,tX)$	$\zeta_{G_2(5)G_2(5)}(5b,7a,tX)$	$\zeta_{G_2(5)}(5c,7a,tX)$
31a	31	806	18600	75950
31b	31	806	18600	75950
31c	31	806	18600	75950
31d	31	806	18600	75950
31e	31	806	18600	75950

tX	$\lvert C_{G_2(5)}(tX)\rvert$	$\zeta_{G_2(5)}(5d,7a,tX)$	$\zeta_{G_2(5)}(5e,7a,tX)$
31a	31	150350	218550
31b	31	150350	218550
31c	31	150350	218550
31d	31	150350	218550
31e	31	150350	218550

Table 43. *Partial structure constants of* $3{\cdot}McL{:}2$.

tX	$\lvert C_{3\cdot McL:2}(tX)\rvert$	$\zeta_{3\cdot McL:2}(5a,7a,tX)$	$\zeta_{3\cdot McL:2}(5b,7a,tX)$
11a	66	57024	1710720
11b	66	57024	1710720

Table 44. *Partial structure constants of* $2{\cdot}A_{11}$.

tX	$\lvert C_{2\cdot A_{11}}(tX)\rvert$	$\zeta_{2\cdot A_{11}}(5a,7a,tX)$	$\zeta_{2\cdot A_{11}}(5b,7a,tX)$
11a	22	0	4026
11b	22	0	4026

5.2. The $(5,11,r)$*-Generations of* Ly

We shall consider all $r \in \{31,37,67\}$. No maximal subgroup has any contribution here since there is none containing elements of orders 31, 37 and 67.

Proposition 5.5. Ly is $(5,11,31)$-generated.

Proof Using Table 45, we have that $\zeta_{Ly}(5A,11A,31A) = \zeta_{Ly}(5A,11A,31B) = \zeta_{Ly}(5A,11A,31C) = \zeta_{Ly}(5A,11A,31D) = \zeta_{Ly}(5A,11A,31E) = 348203997 = \zeta_{Ly}(5A,11B,31A) = \zeta_{Ly}(5A,11B,31B) = \zeta_{Ly}(5A,11B,31C) = \zeta_{Ly}(5A,11B,31D) = \zeta_{Ly}(5A,11B,31E)$ and $\zeta_{Ly}(5B,11A,31A) = \zeta_{Ly}(5B,11A,31B) = \zeta_{Ly}(5B,11A,31C) = \zeta_{Ly}(5B,11A,31D) = \zeta_{Ly}(5B,11A,31E) = 209102876325 = \zeta_{Ly}(5B,11B,31A) = \zeta_{Ly}(5B,11B,31B) = \zeta_{Ly}(5B,11B,31C) = \zeta_{Ly}(5B,11B,31D) = \zeta_{Ly}(5B,11B,31E)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(5A,11A,31A) = \zeta^*_{Ly}(5A,11A,31B) = \zeta^*_{Ly}(5A,11A,31C) = \zeta^*_{Ly}(5A,11A,31D) = \zeta^*_{Ly}(5A,11A,31E) = 348203997 = \zeta^*_{Ly}(5A,11B,31A) = \zeta^*_{Ly}(5A,11B,31B) = \zeta^*_{Ly}(5A,11B,31C) = \zeta^*_{Ly}(5A,11B,31D) = \zeta^*_{Ly}(5A,11B,31E)$ and $\zeta^*_{Ly}(5B,11A,31A) = \zeta^*_{Ly}(5B,11A,31B) = \zeta^*_{Ly}(5B,11A,31C) = \zeta^*_{Ly}(5B,11A,31D) = \zeta^*_{Ly}(5B,11A,31E) = 209102876325 = \zeta^*_{Ly}(5B,11B,31A) = \zeta^*_{Ly}(5B,11B,31B) = \zeta^*_{Ly}(5B,11B,31C) =$

$\zeta^*_{Ly}(5B, 11B, 31D) = \zeta^*_{Ly}(5B, 11B, 31E)$ proving that Ly is $(5, 11, 31)$-generated.

Proposition 5.6. Ly is $(5, 11, 37)$-generated.

Proof By Table 45, we have that $\zeta_{Ly}(5A, 11A, 37A) = \zeta_{Ly}(5A, 11A, 37B) = 351751378 = \zeta_{Ly}(5A, 11B, 37A) = \zeta_{Ly}(5A, 11B, 37B)$ and $\zeta_{Ly}(5B, 11A, 37A) = \zeta_{Ly}(5B, 11A, 37B) = 208778152675 = \zeta_{Ly}(5B, 11B, 37A) = \zeta_{Ly}(5B, 11B, 37B)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(5A, 11A, 37A) = \zeta^*_{Ly}(5A, 11A, 37B) = 351751378 = \zeta^*_{Ly}(5A, 11B, 37A) = \zeta^*_{Ly}(5A, 11B, 37B)$ and $\zeta^*_{Ly}(5B, 11A, 37A) = \zeta^*_{Ly}(5B, 11A, 37B) = 208778152675 = \zeta^*_{Ly}(5B, 11B, 37A) = \zeta^*_{Ly}(5B, 11B, 37B)$ proving that Ly is $(5, 11, 37)$-generated.

Proposition 5.7. Ly is $(5, 11, 67)$-generated.

Proof By Table 45, we get that $\zeta_{Ly}(5A, 11A, 67A) = \zeta_{Ly}(5A, 11A, 67B) = \zeta_{Ly}(5A, 11A, 67C) = 351043686 = \zeta_{Ly}(5A, 11B, 67A) = \zeta_{Ly}(5A, 11B, 67B) = \zeta_{Ly}(5A, 11B, 67C)$ and $\zeta_{Ly}(5B, 11A, 67A) = \zeta_{Ly}(5B, 11A, 67B) = \zeta_{Ly}(5B, 11A, 67C) = 208734902100 = \zeta_{Ly}(5B, 11B, 67A) = \zeta_{Ly}(5B, 11B, 67B) = \zeta_{Ly}(5B, 11B, 67C)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(5A, 11A, 67A) = \zeta^*_{Ly}(5A, 11A, 67B) = \zeta^*_{Ly}(5A, 11A, 67C) = 351043686 = \zeta^*_{Ly}(5A, 11B, 67A) = \zeta^*_{Ly}(5A, 11B, 67B) = \zeta^*_{Ly}(5A, 11B, 67C)$ and $\zeta^*_{Ly}(5B, 11A, 67A) = \zeta^*_{Ly}(5B, 11A, 67B) = \zeta^*_{Ly}(5B, 11A, 67C) = 208734902100 = \zeta^*_{Ly}(5B, 11B, 67A) = \zeta^*_{Ly}(5B, 11B, 67B) = \zeta^*_{Ly}(5B, 11B, 67C)$ proving that Ly is $(5, 11, 67)$-generated.

Table 45. *Partial structure constants of Ly.*

tX	$\|C_{Ly}(tX)\|$	$\zeta_{Ly}(5A, 11A, tX)$	$\zeta_{Ly}(5A, 11B, tX)$	$\zeta_{Ly}(5B, 11A, tX)$	$\zeta_{Ly}(5B, 11B, tX)$
31A	31	348203997	348203997	209102876325	209102876325
31B	31	348203997	348203997	209102876325	209102876325
31C	31	348203997	348203997	209102876325	209102876325
31D	31	348203997	348203997	209102876325	209102876325
31E	31	348203997	348203997	209102876325	209102876325
37A	37	351751378	351751378	208778152675	208778152675
37B	37	351751378	351751378	208778152675	208778152675
67A	67	351043686	351043686	208734902100	208734902100
67B	67	351043686	351043686	208734902100	208734902100
67C	67	351043686	351043686	208734902100	208734902100

5.3. The $(5, 31, r)$-Generations of Ly

We shall consider all $r \in \{37, 67\}$. No maximal subgroup has any contribution here since there is none containing elements of orders 37 and 67.

Proposition 5.8. Ly is $(5, 31, 37)$-generated.

Proof By Table 46, we obtain that $\zeta_{Ly}(5A, 31A, 37A) = \zeta_{Ly}(5A, 31A, 37B) = \zeta_{Ly}(5A, 31B, 37A) = \zeta_{Ly}(5A, 31B, 37B) = \zeta_{Ly}(5A, 31C, 37A) = 741364375 = \zeta_{Ly}(5A, 31C, 37B) = \zeta_{Ly}(5A, 31D, 37A) = \zeta_{Ly}(5A, 31D, 37B) = \zeta_{Ly}(5A, 31E, 37A) = \zeta_{Ly}(5A, 31E, 37B)$ and $\zeta_{Ly}(5B, 31A, 37A) = \zeta_{Ly}(5B, 31A, 37B) = \zeta_{Ly}(5B, 31B, 37A) = \zeta_{Ly}(5B, 31B, 37B) = \zeta_{Ly}(5B, 31C, 37A) = 445187954375 = \zeta_{Ly}(5B, 31C, 37B) = \zeta_{Ly}(5B, 31D, 37A) = \zeta_{Ly}(5B, 31D, 37B) = \zeta_{Ly}(5B, 31E, 37A) = \zeta_{Ly}(5B, 31E, 37B)$. Since no maximal subgroup contributes here, we thus obtain that $\zeta^*_{Ly}(5A, 31A, 37A) = \zeta^*_{Ly}(5A, 31A, 37B) = \zeta^*_{Ly}(5A, 31B, 37A) = \zeta^*_{Ly}(5A, 31B, 37B) = \zeta^*_{Ly}(5A, 31C, 37A) = 741364375 = \zeta^*_{Ly}(5A, 31C, 37B) = \zeta^*_{Ly}(5A, 31D, 37A) = \zeta^*_{Ly}(5A, 31D, 37B) = \zeta^*_{Ly}(5A, 31E, 37A) = \zeta^*_{Ly}(5A, 31E, 37B)$ and $\zeta^*_{Ly}(5B, 31A, 37A) = \zeta^*_{Ly}(5B, 31A, 37B) = \zeta^*_{Ly}(5B, 31B, 37A) = \zeta^*_{Ly}(5B, 31B, 37B) = \zeta^*_{Ly}(5B, 31C, 37A) = 445187954375 = \zeta^*_{Ly}(5B, 31C, 37B) = \zeta^*_{Ly}(5B, 31D, 37A) = \zeta^*_{Ly}(5B, 31D, 37B) = \zeta^*_{Ly}(5B, 31E, 37A) = \zeta^*_{Ly}(5B, 31E, 37B)$ proving that Ly is $(5, 31, 37)$-generated.

Proposition 5.9. Ly is $(5, 31, 67)$-generated.

Proof From Table 46, we get that $\zeta_{Ly}(5A, 31A, 67A) = \zeta_{Ly}(5A, 31A, 67B) = \zeta_{Ly}(5A, 31A, 67C) = \zeta_{Ly}(5A, 31B, 67A) = \zeta_{Ly}(5A, 31B, 67B) = \zeta_{Ly}(5A, 31B, 67C) = \zeta_{Ly}(5A, 31C, 67A) = 742150625 = \zeta_{Ly}(5A, 31C, 67B) = \zeta_{Ly}(5A, 31C, 67C) = \zeta_{Ly}(5A, 31D, 67A) = \zeta_{Ly}(5A, 31D, 67B) = \zeta_{Ly}(5A, 31D, 67C) = \zeta_{Ly}(5A, 31E, 67A) = \zeta_{Ly}(5A, 31E, 67B) = \zeta_{Ly}(5A, 31E, 67C)$ and $\zeta_{Ly}(5B, 31A, 67A) = \zeta_{Ly}(5B, 31A, 67B) = \zeta_{Ly}(5B, 31A, 67C) = \zeta_{Ly}(5B, 31B, 67A) = \zeta_{Ly}(5B, 31B, 67B) = \zeta_{Ly}(5B, 31B, 67C) = \zeta_{Ly}(5B, 31C, 67A) = 445291924375 = \zeta_{Ly}(5B, 31C, 67B) = \zeta_{Ly}(5B, 31C, 67C) = \zeta_{Ly}(5B, 31D, 67A) = \zeta_{Ly}(5B, 31D, 67B) = \zeta_{Ly}(5B, 31D, 67C) = \zeta_{Ly}(5B, 31E, 67A) = \zeta_{Ly}(5B, 31E, 67B) = \zeta_{Ly}(5B, 31E, 67C)$. Since no maximal subgroup contributes here, we obtain that $\zeta^*_{Ly}(5A, 31A, 67A) = \zeta^*_{Ly}(5A, 31A, 67B) = \zeta^*_{Ly}(5A, 31A, 67C) = \zeta^*_{Ly}(5A, 31B, 67A) = \zeta^*_{Ly}(5A, 31B, 67B) = \zeta^*_{Ly}(5A, 31B, 67C) = \zeta^*_{Ly}(5A, 31C, 67A) = 742150625 = \zeta^*_{Ly}(5A, 31C, 67B) = \zeta^*_{Ly}(5A, 31C, 67C) =$

$\zeta^*_{Ly}(5A,31D,67A) = \zeta^*_{Ly}(5A,31D,67B) = \zeta^*_{Ly}(5A,31D,67C) = \zeta^*_{Ly}(5A,31E,67A) = \zeta^*_{Ly}(5A,31E,67B) = \zeta^*_{Ly}(5A,31E,67C)$ and $\zeta^*_{Ly}(5B,31A,67A) = \zeta^*_{Ly}(5B,31A,67B) = \zeta^*_{Ly}(5B,31A,67C) = \zeta^*_{Ly}(5B,31B,67A) = \zeta^*_{Ly}(5B,31B,67B) = \zeta^*_{Ly}(5B,31B,67C) = \zeta^*_{Ly}(5B,31C,67A) = 445291924375 = \zeta^*_{Ly}(5B,31C,67B) = \zeta^*_{Ly}(5B,31C,67C) = \zeta^*_{Ly}(5B,31D,67A) = \zeta^*_{Ly}(5B,31D,67B) = \zeta^*_{Ly}(5B,31D,67C) = \zeta^*_{Ly}(5B,31E,67A) = \zeta^*_{Ly}(5B,31E,67B) = \zeta^*_{Ly}(5B,31E,67C)$ proving that Ly is $(5,31,67)$-generated.

Table 46. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(5A,31A,tX)$	$\zeta_{Ly}(5A,31B,tX)$	$\zeta_{Ly}(5A,31C,tX)$
37A	37	741364375	741364375	741364375
37B	37	741364375	741364375	741364375
67A	67	742150625	742150625	742150625
67B	67	742150625	742150625	742150625
67C	67	742150625	742150625	742150625
tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(5A,31D,tX)$	$\zeta_{Ly}(5A,31E,tX)$	
37A	37	741364375	741364375	
37B	37	741364375	741364375	
67A	67	742150625	742150625	
67B	67	742150625	742150625	
67C	67	742150625	742150625	
tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(5B,31A,tX)$	$\zeta_{Ly}(5B,31B,tX)$	$\zeta_{Ly}(5B,31C,tX)$
37A	37	445187954375	445187954375	445187954375
37B	37	445187954375	445187954375	445187954375
67A	67	445291924375	445291924375	445291924375
67B	67	445291924375	445291924375	445291924375
67C	67	445291924375	445291924375	445291924375
tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(5B,31D,tX)$	$\zeta_{Ly}(5B,31E,tX)$	
37A	37	445187954375	445187954375	
37B	37	445187954375	445187954375	
67A	67	445291924375	445291924375	
67B	67	445291924375	445291924375	
67C	67	445291924375	445291924375	

5.4. The $(5,37,r)$-Generations of Ly

We shall only consider $r = 67$ here. No maximal subgroup has any contribution here since there is none containing elements of order 67.

Proposition 5.10. Ly is $(5,37,67)$-generated.

Proof According to Table 47, $\zeta_{Ly}(5A,37A,67A) = \zeta_{Ly}(5A,37A,67B) = \zeta_{Ly}(5A,37A,67C) = 611815123 = \zeta_{Ly}(5A,37B,67A) = \zeta_{Ly}(5A,37B,67B) = \zeta_{Ly}(5A,37B,67C)$ and $\zeta_{Ly}(5B,37A,67A) = \zeta_{Ly}(5B,37A,67B) = \zeta_{Ly}(5B,37A,67C) = 374586995225 = \zeta_{Ly}(5B,37B,67A) = \zeta_{Ly}(5B,37B,67B) = \zeta_{Ly}(5B,37B,67C)$. Since no maximal subgroup contributes here, we obtain that $\zeta^*_{Ly}(5A,37A,67A) = \zeta^*_{Ly}(5A,37A,67B) = \zeta^*_{Ly}(5A,37A,67C) = 611815123 = \zeta^*_{Ly}(5A,37B,67A) = \zeta^*_{Ly}(5A,37B,67B) = \zeta^*_{Ly}(5A,37B,67C)$ and $\zeta^*_{Ly}(5B,37A,67A) = \zeta^*_{Ly}(5B,37A,67B) = \zeta^*_{Ly}(5B,37A,67C) = 374586995225 = \zeta^*_{Ly}(5B,37B,67A) = \zeta^*_{Ly}(5B,37B,67B) = \zeta^*_{Ly}(5B,37B,67C)$ proving that Ly is $(5,37,67)$-generated.

Table 47. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(5A,37A,tX)$	$\zeta_{Ly}(5A,37B,tX)$	$\zeta_{Ly}(5B,37A,tX)$	$\zeta_{Ly}(5B,37B,tX)$
67A	67	611815123	611815123	374586995225	374586995225
67B	67	611815123	611815123	374586995225	374586995225
67C	67	611815123	611815123	374586995225	374586995225

6. The $(7, q, r)$-Generations of Ly

We shall consider all $q \in \{11, 31, 37\}, r \in \{31, 37, 67\}$.

6.1. The $(7, 11, r)$-Generations of Ly

We shall consider all $r \in \{31, 37, 67\}$ but we notice that no maximal subgroup has any contribution here since there is none containing elements of orders 31, 37 and 67.

Proposition 6.1. Ly is $(7, 11, 31)$-generated.

Proof By Table 48, we have that $\zeta_{Ly}(7A, 11A, 31A) = \zeta_{Ly}(7A, 11A, 31B) = \zeta_{Ly}(7A, 11A, 31C) = \zeta_{Ly}(7A, 11A, 31D) = \zeta_{Ly}(7A, 11A, 31E) = 4668576991056 = \zeta_{Ly}(7A, 11B, 31A) = \zeta_{Ly}(7A, 11B, 31B) = \zeta_{Ly}(7A, 11B, 31C) = \zeta_{Ly}(7A, 11B, 31D) = \zeta_{Ly}(7A, 11B, 31E)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(7A, 11A, 31A) = \zeta^*_{Ly}(7A, 11A, 31B) = \zeta^*_{Ly}(7A, 11A, 31C) = \zeta^*_{Ly}(7A, 11A, 31D) = \zeta^*_{Ly}(7A, 11A, 31E) = 4668576991056 = \zeta^*_{Ly}(7A, 11B, 31A) = \zeta^*_{Ly}(7A, 11B, 31B) = \zeta^*_{Ly}(7A, 11B, 31C) = \zeta^*_{Ly}(7A, 11B, 31D) = \zeta^*_{Ly}(7A, 11B, 31E)$ proving that Ly is $(7, 11, 31)$-generated.

Proposition 6.2. Ly is $(7, 11, 37)$-generated.

Proof By Table 48, we have that $\zeta_{Ly}(7A, 11A, 37A) = \zeta_{Ly}(7A, 11B, 37A) = 46648066712182 = \zeta_{Ly}(7A, 11A, 37B) = \zeta_{Ly}(7A, 11B, 37B)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(7A, 11A, 37A) = \zeta^*_{Ly}(7A, 11B, 37A) = 46648066712182 = \zeta^*_{Ly}(7A, 11A, 37B) = \zeta^*_{Ly}(7A, 11B, 37B)$ proving that Ly is $(7, 11, 37)$-generated.

Proposition 6.3. Ly is $(7, 11, 67)$-generated.

Proof By Table 48, we have that $\zeta_{Ly}(7A, 11A, 67A) = \zeta_{Ly}(7A, 11A, 67B) = \zeta_{Ly}(7A, 11A, 67C) = 4664812116078 = \zeta_{Ly}(7A, 11B, 67A) = \zeta_{Ly}(7A, 11B, 67B) = \zeta_{Ly}(7A, 11B, 67C)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(7A, 11A, 67A) = \zeta^*_{Ly}(7A, 11A, 67B) = \zeta^*_{Ly}(7A, 11A, 67C) = 4664812116078 = \zeta^*_{Ly}(7A, 11B, 67A) = \zeta^*_{Ly}(7A, 11B, 67B) = \zeta^*_{Ly}(7A, 11B, 67C)$ proving that Ly is $(7, 11, 67)$-generated.

Table 48. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(7A, 11A, tX)$	$\zeta_{Ly}(7A, 11B, tX)$
31A	31	4668576991056	4668576991056
31B	31	4668576991056	4668576991056
31C	31	4668576991056	4668576991056
31D	31	4668576991056	4668576991056
31E	31	4668576991056	4668576991056
37A	37	46648066712182	46648066712182
37B	37	46648066712182	46648066712182
67A	67	4664812116078	4664812116078
67B	67	4664812116078	4664812116078
67C	67	4664812116078	4664812116078

6.2. The $(7, 31, r)$-Generations of Ly

We shall consider all $r \in \{37, 67\}$ here. No maximal subgroup has any contribution here since there is none containing elements of orders 37 and 67.

Proposition 6.4. Ly is $(7, 31, 37)$-generated.

Proof By Table 49, we have that $\zeta_{Ly}(7A, 31A, 37A) = \zeta_{Ly}(7A, 31B, 37A) = \zeta_{Ly}(7A, 31C, 37A) = \zeta_{Ly}(7A, 31D, 37A) = \zeta_{Ly}(7A, 31E, 37A) = 9939550500000 = \zeta_{Ly}(7A, 31A, 37B) = \zeta_{Ly}(7A, 31B, 37B) = \zeta_{Ly}(7A, 31C, 37B) = \zeta_{Ly}(7A, 31D, 37B) = \zeta_{Ly}(7A, 31E, 37B)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(7A, 31A, 37A) = \zeta^*_{Ly}(7A, 31B, 37A) = \zeta^*_{Ly}(7A, 31C, 37A) = \zeta^*_{Ly}(7A, 31D, 37A) = \zeta^*_{Ly}(7A, 31E, 37A) = 9939550500000 = \zeta^*_{Ly}(7A, 31A, 37B) = \zeta^*_{Ly}(7A, 31B, 37B) = \zeta^*_{Ly}(7A, 31C, 37B) = \zeta^*_{Ly}(7A, 31D, 37B) = \zeta^*_{Ly}(7A, 31E, 37B)$ proving that Ly is $(7, 31, 37)$-generated.

Proposition 6.5. Ly is $(7, 31, 67)$-generated.

Proof By Table 49, we have that $\zeta_{Ly}(7A, 31A, 67A) = \zeta_{Ly}(7A, 31A, 67B) = \zeta_{Ly}(7A, 31A, 67C) = \zeta_{Ly}(7A, 31B, 67A) = \zeta_{Ly}(7A, 31B, 67B) = \zeta_{Ly}(7A, 31B, 67C) = \zeta_{Ly}(7A, 31C, 67A) = 9939550500000 = \zeta_{Ly}(7A, 31C, 67B) = \zeta_{Ly}(A, 31C, 67C) = \zeta_{Ly}(7A, 31D, 67A) = \zeta_{Ly}(7A, 31D, 67B) = \zeta_{Ly}(7A, 31D, 67C) = \zeta_{Ly}(7A, 31E, 67A) = \zeta_{Ly}(7A, 31E, 67B) = \zeta_{Ly}(7A, 31E, 67C)$. Since there is no contribution from any of the maximal subgroups, we have that $\zeta^*_{Ly}(7A, 31A, 67A) = \zeta^*_{Ly}(7A, 31A, 67B) = \zeta^*_{Ly}(7A, 31A, 67C) = \zeta^*_{Ly}(7A, 31B, 67A) = \zeta^*_{Ly}(7A, 31B, 67B) = \zeta^*_{Ly}(7A, 31B, 67C) = \zeta^*_{Ly}(7A, 31C, 67A) = 9939550500000 = \zeta^*_{Ly}(7A, 31C, 67B) = \zeta^*_{Ly}(A, 31C, 67C) = \zeta^*_{Ly}(7A, 31D, 67A) = \zeta^*_{Ly}(7A, 31D, 67B) = \zeta^*_{Ly}(7A, 31D, 67C) = \zeta^*_{Ly}(7A, 31E, 67A) = \zeta^*_{Ly}(7A, 31E, 67B) = \zeta^*_{Ly}(7A, 31E, 67C)$ proving that Ly is $(7, 31, 67)$-generated.

Table 49. *Partial structure constants of* Ly.

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(7A, 31A, tX)$	$\zeta_{Ly}(7A, 31B, tX)$	$\zeta_{Ly}(7A, 31C, tX)$
37A	37	9939550500000	9939550500000	9939550500000
37B	37	9939550500000	9939550500000	9939550500000
67A	67	9939550500000	9939550500000	9939550500000
67B	67	9939550500000	9939550500000	9939550500000
67C	67	9939550500000	9939550500000	9939550500000

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(7A, 31D, tX)$	$\zeta_{Ly}(7A, 31E, tX)$
37A	37	9939550500000	9939550500000
37B	37	9939550500000	9939550500000
67A	67	9939550500000	9939550500000
67B	67	9939550500000	9939550500000
67C	67	9939550500000	9939550500000

6.3. The $(7, 37, r)$-Generations of Ly

We shall only consider $r = 67$. However no maximal subgroup has any contribution here since there is none that contains elements of order 67.

Proposition 6.6. Ly is $(7, 37, 67)$-generated.

Proof By Table 50, we have that $\zeta_{Ly}(7A, 37A, 67A) = \zeta_{Ly}(7A, 37A, 67B) = \zeta_{Ly}(7A, 37A, 67C) = 8341173788256 = \zeta_{Ly}(7A, 37B, 67A) = \zeta_{Ly}(7A, 37B, 67B) = \zeta_{Ly}(7A, 37B, 67C)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(7A, 37A, 67A) = \zeta^*_{Ly}(7A, 37A, 67B) = \zeta^*_{Ly}(7A, 37A, 67C) = 8341173788256 = \zeta^*_{Ly}(7A, 37B, 67A) = \zeta^*_{Ly}(7A, 37B, 67B) = \zeta^*_{Ly}(7A, 37B, 67C)$ proving that Ly is $(7, 37, 67)$-generated.

Table 50. *Partial structure constants of* Ly.

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(7A, 37A, tX)$	$\zeta_{Ly}(7A, 37B, tX)$
67A	67	8341173788256	8341173788256
67B	67	8341173788256	8341173788256
67C	67	8341173788256	8341173788256

7. The $(11, q, r)$-Generations of Ly

We shall consider here all $q \in \{31, 37\}, r \in \{37, 67\}$.

7.1. The $(11, 31, r)$-Generations of Ly

We shall consider here all $r \in \{37, 67\}$. However we observe that no maximal subgroup has any contribution here since there is none containing elements of orders 37 and 67.

Proposition 7.1. Ly is $(11, 31, 37)$-generated.

Proof By Table 51, we have that $\zeta_{Ly}(11A, 31A, 37A) = \zeta_{Ly}(11A, 31B, 37A) = \zeta_{Ly}(11A, 31C, 37A) = \zeta_{Ly}(11A, 31D, 37A) = \zeta_{Ly}(11A, 31E, 37A) = \zeta_{Ly}(11A, 31A, 37B) = \zeta_{Ly}(11A, 31B, 37B) = \zeta_{Ly}(11A, 31C, 37B) = \zeta_{Ly}(11A, 31D, 37B) = \zeta_{Ly}(11A, 31E, 37B) = 25301095453125 = \zeta_{Ly}(11B, 31A, 37A) = \zeta_{Ly}(11B, 31B, 37A) = \zeta_{Ly}(11B, 31C, 37A) = \zeta_{Ly}(11B, 31D, 37A) = \zeta_{Ly}(11B, 31E, 37A) = \zeta_{Ly}(11B, 31A, 37B) = \zeta_{Ly}(11B, 31B, 37B) = \zeta_{Ly}(11B, 31C, 37B) = \zeta_{Ly}(11B, 31D, 37B) = \zeta_{Ly}(11B, 31E, 37B)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(11A, 31A, 37A) = \zeta^*_{Ly}(11A, 31B, 37A) = \zeta^*_{Ly}(11A, 31C, 37A) = \zeta^*_{Ly}(11A, 31D, 37A) = \zeta^*_{Ly}(11A, 31E, 37A) = \zeta^*_{Ly}(11A, 31A, 37B) = \zeta^*_{Ly}(11A, 31B, 37B) = \zeta^*_{Ly}(11A, 31C, 37B) = \zeta^*_{Ly}(11A, 31D, 37B) = \zeta^*_{Ly}(11A, 31E, 37B) = 25301095453125 = \zeta^*_{Ly}(11B, 31A, 37A) = \zeta^*_{Ly}(11B, 31B, 37A) = \zeta^*_{Ly}(11B, 31C, 37A) = \zeta^*_{Ly}(11B, 31D, 37A) = \zeta^*_{Ly}(11B, 31E, 37A) = \zeta^*_{Ly}(11B, 31A, 37B) = \zeta^*_{Ly}(11B, 31B, 37B) = \zeta^*_{Ly}(11B, 31C, 37B) = \zeta^*_{Ly}(11B, 31D, 37B) = \zeta^*_{Ly}(11B, 31E, 37B)$ proving that Ly is $(11, 31, 37)$-generated.

Proposition 7.2. Ly is $(11, 31, 67)$-generated.

Proof By Table 51, we have that $\zeta_{Ly}(11A, 31A, 67A) = \zeta_{Ly}(11A, 31A, 67B) = \zeta_{Ly}(11A, 31A, 67C) = \zeta_{Ly}(11A, 31B, 67A) = \zeta_{Ly}(11A, 31B, 67B) = \zeta_{Ly}(11A, 31B, 67C) = \zeta_{Ly}(11A, 31C, 67A) = \zeta_{Ly}(11A, 31C, 67B) = \zeta_{Ly}(11A, 31C, 67C) = \zeta_{Ly}(11A, 31D, 67A) = \zeta_{Ly}(11A, 31D, 67B) = \zeta_{Ly}(11A, 31D, 67C) = \zeta_{Ly}(11A, 31E, 67A) = \zeta_{Ly}(11A, 31E, 67B) = \zeta_{Ly}(11A, 31E, 67C) = 25300674000000 = \zeta_{Ly}(11B, 31A, 67A) = \zeta_{Ly}(11B, 31A, 67B) = \zeta_{Ly}(11B, 31A, 67C) = \zeta_{Ly}(11B, 31B, 67A) = \zeta_{Ly}(11B, 31B, 67B) = \zeta_{Ly}(11B, 31B, 67C) = \zeta_{Ly}(11B, 31C, 67A) = \zeta_{Ly}(11B, 31C, 67B) = \zeta_{Ly}(11B, 31C, 67C) = \zeta_{Ly}(11B, 31D, 67A) = \zeta_{Ly}(11B, 31D, 67B) = \zeta_{Ly}(11B, 31D, 67C) = \zeta_{Ly}(11B, 31E, 67A) =$

$\zeta_{Ly}(11B,31E,67B) = \zeta_{Ly}(11B,31E,67C)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(11A,31A,67A) = \zeta^*_{Ly}(11A,31A,67B) = \zeta^*_{Ly}(11A,31A,67C) = \zeta^*_{Ly}(11A,31B,67A) = \zeta^*_{Ly}(11A,31B,67B) = \zeta^*_{Ly}(11A,31B,67C) = \zeta^*_{Ly}(11A,31C,67A) = \zeta^*_{Ly}(11A,31C,67B) = \zeta^*_{Ly}(11A,31C,67C) = \zeta^*_{Ly}(11A,31D,67A) = \zeta^*_{Ly}(11A,31D,67B) = \zeta^*_{Ly}(11A,31D,67C) = \zeta^*_{Ly}(11A,31E,67A) = \zeta^*_{Ly}(11A,31E,67B) = \zeta^*_{Ly}(11A,31E,67C) = 25300674000000 = \zeta^*_{Ly}(11B,31A,67A) = \zeta^*_{Ly}(11B,31A,67B) = \zeta^*_{Ly}(11B,31A,67C) = \zeta^*_{Ly}(11B,31B,67A) = \zeta^*_{Ly}(11B,31B,67B) = \zeta^*_{Ly}(11B,31B,67C) = \zeta^*_{Ly}(11B,31C,67A) = \zeta^*_{Ly}(11B,31C,67B) = \zeta^*_{Ly}(11B,31C,67C) = \zeta^*_{Ly}(11B,31D,67A) = \zeta^*_{Ly}(11B,31D,67B) = \zeta^*_{Ly}(11B,31D,67C) = \zeta^*_{Ly}(11B,31E,67A) = \zeta^*_{Ly}(11B,31E,67B) = \zeta^*_{Ly}(11B,31E,67C)$ proving that Ly is $(11,31,67)$-generated.

Table 51. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(11A,31A,tX)$	$\zeta_{Ly}(11A,31B,tX)$	$\zeta_{Ly}(11A,31C,tX)$	$\zeta_{Ly}(11A,31D,tX)$
$37A$	37	25301095453125	25301095453125	25301095453125	25301095453125
$37B$	37	25301095453125	25301095453125	25301095453125	25301095453125
$67A$	67	25300674000000	25300674000000	25300674000000	25300674000000
$67B$	67	25300674000000	25300674000000	25300674000000	25300674000000
$67C$	67	25300674000000	25300674000000	25300674000000	25300674000000

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(11A,31E,tX)$
$37A$	37	25301095453125
$37B$	37	25301095453125
$67A$	67	25300674000000
$67B$	67	25300674000000
$67C$	67	25300674000000

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(11B,31A,tX)$	$\zeta_{Ly}(11B,31B,tX)$	$\zeta_{Ly}(11B,31C,tX)$	$\zeta_{Ly}(11B,31D,tX)$
$37A$	37	25301095453125	25301095453125	25301095453125	25301095453125
$37B$	37	25301095453125	25301095453125	25301095453125	25301095453125
$67A$	67	25300674000000	25300674000000	25300674000000	25300674000000
$67B$	67	25300674000000	25300674000000	25300674000000	25300674000000
$67C$	67	25300674000000	25300674000000	25300674000000	25300674000000

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(11B,31E,tX)$
$37A$	37	25301095453125
$37B$	37	25301095453125
$67A$	67	25300674000000
$67B$	67	25300674000000
$67C$	67	25300674000000

7.2. The $(11,37,67)$-Generations of Ly

We observe that no maximal subgroup has any contribution here since there is none containing elements of order 67.

Proposition 7.3. Ly is $(11,37,67)$-generated.

Proof By Table 52, we have that $\zeta_{Ly}(11A,37A,67A) = \zeta_{Ly}(11A,37A,67B) = \zeta_{Ly}(11A,37A,67C) = \zeta_{Ly}(11A,37B,67A) = \zeta_{Ly}(11A,37B,67B) = \zeta_{Ly}(11A,37B,67C) = 21189314475000 = \zeta_{Ly}(11B,37A,67A) = \zeta_{Ly}(11B,37A,67B) = \zeta_{Ly}(11B,37A,67C) = \zeta_{Ly}(11B,37B,67A) = \zeta_{Ly}(11B,37B,67B) = \zeta_{Ly}(11B,37B,67C)$. Since there is no contribution from any of the maximal subgroups, we thus obtain that $\zeta^*_{Ly}(11A,37A,67A) = \zeta^*_{Ly}(11A,37A,67B) = \zeta^*_{Ly}(11A,37A,67C) = \zeta^*_{Ly}(11A,37B,67A) = \zeta^*_{Ly}(11A,37B,67B) = \zeta^*_{Ly}(11A,37B,67C) = 21189314475000 = \zeta^*_{Ly}(11B,37A,67A) = \zeta^*_{Ly}(11B,37A,67B) = \zeta^*_{Ly}(11B,37A,67C) = \zeta^*_{Ly}(11B,37B,67A) = \zeta^*_{Ly}(11B,37B,67B) = \zeta^*_{Ly}(11B,37B,67C)$ proving that Ly is $(11,37,67)$-generated.

Table 52. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(11A,37A,tX)$	$\zeta_{Ly}(11A,37B,tX)$	$\zeta_{Ly}(11B,37A,tX)$	$\zeta_{Ly}(11B,37B,tX)$
67A	67	21189314475000	21189314475000	21189314475000	21189314475000
67B	67	21189314475000	21189314475000	21189314475000	21189314475000
67C	67	21189314475000	21189314475000	21189314475000	21189314475000

8. The $(31, q, r)$-Generations of Ly

For the $(31, q, r)$-generations of Ly, we shall consider $q = 37, r = 67$ only.

8.1. The $(31, 37, 67)$-Generations of Ly

We observe that no maximal subgroup has any contribution here since there is none containing elements of order 67.

Proposition 8.1. Ly is $(31, 37, 67)$-generated.

Proof By Table 53, we have that $\zeta_{Ly}(31A, 37A, 67A) = \zeta_{Ly}(31A, 37A, 67B) = \zeta_{Ly}(31A, 37A, 67C) = \zeta_{Ly}(31B, 37A, 67A) = \zeta_{Ly}(31B, 37A, 67B) = \zeta_{Ly}(31B, 37A, 67C) = \zeta_{Ly}(31C, 37A, 67A) = \zeta_{Ly}(31C, 37A, 67B) = \zeta_{Ly}(31C, 37A, 67C) = \zeta_{Ly}(31E, 37A, 67A) = \zeta_{Ly}(31E, 37A, 67B) = \zeta_{Ly}(31E, 37A, 67C) = 45130930953125 = \zeta_{Ly}(31A, 37B, 67A) = \zeta_{Ly}(31A, 37B, 67B) = \zeta_{Ly}(31A, 37B, 67C) = \zeta_{Ly}(31B, 37B, 67A) = \zeta_{Ly}(31B, 37B, 67B) = \zeta_{Ly}(31B, 37B, 67C) = \zeta_{Ly}(31C, 37B, 67A) = \zeta_{Ly}(31C, 37B, 67B) = \zeta_{Ly}(31C, 37B, 67C) = \zeta_{Ly}(31E, 37B, 67A) = \zeta_{Ly}(31E, 37B, 67B) = \zeta_{Ly}(31E, 37B, 67C)$ and $\zeta_{Ly}(31D, 37A, 67A) = \zeta_{Ly}(31D, 37A, 67B) = \zeta_{Ly}(31D, 37A, 67C) = 53865046312500 = \zeta_{Ly}(31D, 37B, 67A) = \zeta_{Ly}(31D, 37B, 67B) = \zeta_{Ly}(31D, 37B, 67C)$. Since there is no contribution from any of the maximal subgroups, we obtain that $\zeta^*_{Ly}(31A, 37A, 67A) = \zeta^*_{Ly}(31A, 37A, 67B) = \zeta^*_{Ly}(31A, 37A, 67C) = \zeta^*_{Ly}(31B, 37A, 67A) = \zeta^*_{Ly}(31B, 37A, 67B) = \zeta^*_{Ly}(31B, 37A, 67C) = \zeta^*_{Ly}(31C, 37A, 67A) = \zeta^*_{Ly}(31C, 37A, 67B) = \zeta^*_{Ly}(31C, 37A, 67C) = \zeta^*_{Ly}(31E, 37A, 67A) = \zeta^*_{Ly}(31E, 37A, 67B) = \zeta^*_{Ly}(31E, 37A, 67C) = 45130930953125 = \zeta^*_{Ly}(31A, 37B, 67A) = \zeta^*_{Ly}(31A, 37B, 67B) = \zeta^*_{Ly}(31A, 37B, 67C) = \zeta^*_{Ly}(31B, 37B, 67A) = \zeta^*_{Ly}(31B, 37B, 67B) = \zeta^*_{Ly}(31B, 37B, 67C) = \zeta^*_{Ly}(31C, 37B, 67A) = \zeta^*_{Ly}(31C, 37B, 67B) = \zeta^*_{Ly}(31C, 37B, 67C) = \zeta^*_{Ly}(31E, 37B, 67A) = \zeta^*_{Ly}(31E, 37B, 67B) = \zeta^*_{Ly}(31E, 37B, 67C)$ and $\zeta^*_{Ly}(31D, 37A, 67A) = \zeta^*_{Ly}(31D, 37A, 67B) = \zeta^*_{Ly}(31D, 37A, 67C) = 53865046312500 = \zeta^*_{Ly}(31D, 37B, 67A) = \zeta^*_{Ly}(31D, 37B, 67B) = \zeta^*_{Ly}(31D, 37B, 67C)$ proving that Ly is $(31, 37, 67)$-generated.

Table 53. *Partial structure constants of Ly.*

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(31A, 37A, tX)$	$\zeta_{Ly}(31B, 37A, tX)$	$\zeta_{Ly}(31C, 37A, tX)$
67A	67	45130930953125	45130930953125	45130930953125
67B	67	45130930953125	45130930953125	45130930953125
67C	67	45130930953125	45130930953125	45130930953125

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(31D, 37A, tX)$	$\zeta_{Ly}(31E, 37A, tX)$
67A	67	53865046312500	45130930953125
67B	67	53865046312500	45130930953125
67C	67	53865046312500	45130930953125

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(31A, 37B, tX)$	$\zeta_{Ly}(31B, 37B, tX)$	$\zeta_{Ly}(31C, 37B, tX)$
67A	67	45130930953125	45130930953125	45130930953125
67B	67	45130930953125	45130930953125	45130930953125
67C	67	45130930953125	45130930953125	45130930953125

tX	$\lvert C_{Ly}(tX)\rvert$	$\zeta_{Ly}(31D, 37B, tX)$	$\zeta_{Ly}(31E, 37B, tX)$
67A	67	53865046312500	45130930953125
67B	67	53865046312500	45130930953125
67C	67	53865046312500	45130930953125

9. Conclusion

Every finite nonabelian simple group can be generated by a minimum of two of its elements. According to [10], given a finite nonabelian simple group G with ℓ, m, n dividing the order of G such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, does it follow that G is (ℓ, m, n)-generated?

In the current article, we studied and determined the various pairs of elements of Ly from distinct conjugacy classes of elements of distinct prime orders which generate Ly and this study is not in any way exhausted.

Also the study of various combinations of three, four, five etc elements from distinct conjugacy classes which can generate Ly can still be undertaken. The most challenging and possibly daunting study though would be to find the maximum number of elements of Ly from distinct conjugacy classes of its elements which can generate Ly.

As to whether the technique of structure constants which was used in the present study would suffice for the determination of the maximum number of elements of Ly which can generate Ly, still remains to be seen.

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References

[1] F. Ali, *On the ranks of O'N and Ly*, Discrete Appl. Math. 155, (2007), 394-399.

[2] M. D. E. Conder, R. A. Wilson and A. J. Woldar, *The symmetric genus of sporadic groups*, Proc. Amer. Math. Soc. 116 (3), (1992), 653-663.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson *Atlas of Finite Groups*, Oxford University Press, New York, (1985).

[4] S. Ganief and J. Moori, $(2, 3, t)$*-generations for the Janko group* J_3, Comm. Algebra, 23 (12), (1995), 4427-4437.

[5] S. Ganief and J. Moori, (p, q, r)*-generations for the smallest Conway group* Co_3, J. Algebra, 188, (1997), 516-530.

[6] M. S. Ganief, *2-Generations of the Sporadic Simple Groups*, Ph.D Thesis, University on Natal, (1997).

[7] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.10; (2007). (http://www.gap-system.org)

[8] J. F. Hurley and A. Rudvalis, *Finite simple groups*, Amer. Math. Monthly 84, (1977), 693-714.

[9] C. Jansen, K. Lux, R. Parker and R. Wilson *An Atlas of Brauer Characters*, Oxford University Press Inc., New York, (1995).

[10] J. Moori, (p, q, r)*-generations for the Janko groups* J_1 *and* J_2, Nova J. Algebra Geom. 2 (3), (1993), 277-285.

[11] J. Moori, *On the ranks of Janko groups* J_1, J_2, J_3, article presented at the 41st Annual Congress of the South African Mathematical Society, Rand Afrikaans University, Johannesburg, (1998).

[12] M. J. Motalane, *Triple Generations of the Lyons Sporadic Simple Group*, MSc Dissertation, University of South Africa, (2015).

[13] Z Mpono, *The Conjugacy Class Ranks of* M_{24}, International J. Group Theory 6 (4), (2017), 53-58.

[14] Z Mpono, *Triple Generations and Connected Components of Brauer Graphs in* M_{24}, Southeast Asian Bull. Math., 41 (2017), 65-89.

[15] R. Lyons, *Evidence for a new finite simple group*, J. Algebra 20, (1972), 540-569.

[16] T. T. Seretlo, *Fischer Clifford Matrices and Character Tables of Certain Groups Associated with Simple Groups* $O_{10}^{+}(2), HS$ *and* Ly, PhD Thesis, University of KwaZulu-Natal, (2011).

[17] A. J. Woldar, *Representing* M_{11}, M_{12}, M_{22} *and* M_{23} *on surfaces of least genus*, Comm. Algebra 12 (1), 1990, 15-86.

[18] A. J. Woldar, *3/2-generarion of the sporadic simple groups*, Comm. Algebra, 22 (2), (1994), 675-685.