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Q-borderenergeticity Under the Graph Operation of Complements

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Abstract: Let $A(G)$ be the adjacency matrix of graph G. Suppose $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ are the eigenvalues of $A(G)$. The energy of a graph G is denoted by $\varepsilon(G)$, which is defined as the sum of absolute values of its eigenvalues. It is well known that graph energy is found that there are many applications in chemistry. Nikiforov showed that almost all graphs have an energy asymptotically equal to $O(n^{1.5})$. So, almost all graphs are supperenergetic, i.e., their graph energies are more than those of complete graphs with the same orders. This made an end to the study of supperenergetic graphs. Then the concept of a *borderenergetic* graph is proposed by Gutman et al. in 2015. If a graph G of order n satisfies it energy $\varepsilon(G) = 2(n-1)$, then G is called a borderenergetic graph. Recently, Tao and Hou extend this concept to *signless Laplacian energy*. That is, a graph of order n is called *Q-borderenergetic* graph if its signless Laplacian energy is equal to that of the complete graph K_n. In this work, by using the graph operation of complements, we find that, for most of Q-borderenergetic graphs, it can not satisfy themselves and their complements are all Q-borderenergetic. Besides, a new lower bound on signless Laplacian energy of the complement of a Q-borderenergetic graph is established.

Keywords: Signless Laplacian Energy, Q-borderenergetic Graphs, Zagreb Index, Complement

1. Introduction

All graphs in this paper we consider are simple and undirected. Let G be a graph whose order is n and size is m, denoted vertex set and edge set by $V(G)$ and $E(G)$. The complete graph of order n denoted by K_n. For more terminologies and notations, we refer to J. A. Bondy et al. [2].

Let $A(G)$ be the adjacency matrix of graph G. Suppose $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ are the eigenvalues of $A(G)$. The diagonal matrix of the vertex degree of G is $D(G)$. Denoted by $L(G) = D(G) - A(G)$ is the Laplacian matrix of G and $Q(G) = D(G) + A(G)$ is signless Laplacian matrix of G. Assume $\mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ is the eigenvalues of $L(G)$ and $q_n \leq q_{n-1} \leq \cdots \leq q_1$ is the eigenvalues of $Q(G)$.

The energy of a graph G [16] is denoted by $\varepsilon(G)$, which is defined as follow.

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

More knowledge about graph energy and its applications can be referred to the references [3, 15, 18, 20, 22]. Laplacian energy and signless Laplace energy of graph G are denoted by $LE(G)$ and $QE(G)$, respectively. That is, $LE(G) = \sum_{i=1}^{n} |\mu_i - \bar{d}|$ and $QE(G) = \sum_{i=1}^{n} |q_i - \bar{d}|$, where $\bar{d}$ is the average degree of G. For addition information on Laplacian energy and its generalization, we refer to the references [5, 23, 27].

Recently, the concept of a *borderenergetic graph* is proposed by Gutman et al [17]. If a graph G of order n satisfies $\varepsilon(G) = 2(n-1)$, then G is called a borderenergetic *graph*. We can find more researches on borderenergetic *graphs* in the references [8, 9, 13, 19, 24].

For the Laplace energy of a graph G, if it satisfies $LE(G) = LE(K_n) = 2(n-1)$, then G is called *L-borderenergetic* [26]. Related researches on L-borderenergetic graphs can be seen in the references [7, 11, 12, 21].

Similarly, Hou and Tao [25] extended it to the signless Laplacian energy of a graph. For a graph G, if $QE(G) =$

$QE(K_n) = 2(n-1)$ holds, then G is *Q-borderenergetic.* Related researches on signless Laplacian energy of a graph and Q-borderenergetic graphs can be seen in the references [4, 6, 10, 14]. Recently, Deng and Chang et al. [6] construct regular *Q-borderenergetic graphs* by using a regular *Q-borderenergetic graph*. Deng and Li et al. [9, 21] study the complementary graphs of *borderenergetic graphs* and *L-borderenergetic graphs.*

Moreover, in this paper, we focus on the relationship between the Q-borderenergetic *graphs* and their complements. And we find that, for most of Q-borderenergetic graphs which satisfy the bounds of Zagreb index, they and their complements cannot both be Q-borderenergetic. Besides, new bounds of the complement of a Q-borderenergetic graph are given.

2. Main Results

Let G be a connected graph of order n with m edges with having the firt Zagreb index $M_1(G) = \sum_{i=1}^{n} {d_i}^2$, where d_i is the degree of vertex v_i of graph G, where $i = 1,2,..,n$. Let $\overline{G}$ is the complement of G. Assume $\overline{d_i}$ is the degree of v_i of $\overline{G}$, $i = 1,2,..,n$. Let $\overline{m}$ be the size of $\overline{G}$. It's easy to get the first Zagreb index of $\overline{G}$ is that

$$\begin{aligned} M_1(\overline{G}) &= \sum_{i=1}^{n} {\overline{d}_i}^2 = \sum_{i=1}^{n}(n-1-d_i)^2 = \sum_{i=1}^{n}\left[(n-1)^2 - 2(n-1)d_i + {d_i}^2\right] \\ &= (n-1)^2 - 2(n-1)\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} {d_i}^2 \\ &= n(n-1)^2 - 4(n-1)m + M_1(G) \end{aligned} \quad (1)$$

An upper bound of $QE(G)$ is given as follow.

Lemma 1 [1]. If G is a connected (n, m)-graph, then

$$QE(G) \le \varepsilon(G) + \sqrt{nM_1(G) - 4m^2}, \quad (2)$$

where the equality of inequality (2) holds if and only if G is regular.

Corollary 2. Let G be a connected (n, m)-graph. If the complement $\overline{G}$ of G is connected, then

$$QE(\overline{G}) \le \varepsilon(\overline{G}) + \sqrt{nM_1(G) - 4m^2}, \quad (3)$$

where the equality above holds if and only if $\overline{G}$ is regular.

Proof By Lemma 1, it's easy to get

$$QE(\overline{G}) \le \varepsilon(\overline{G}) + \sqrt{nM_1(\overline{G}) - 4\overline{m}^2}, \quad (4)$$

where $\overline{m} = \frac{1}{2}n(n-1) - m$. Put $\overline{m}$ and (1) into (4). Then

$$\begin{aligned} QE(\overline{G}) &\le \varepsilon(\overline{G}) + \sqrt{n[n(n-1)^2 - 4(n-1)m + M_1(G)] - 4\left[\frac{1}{2}n(n-1) - m\right]^2} \\ &= \varepsilon(\overline{G}) + \sqrt{n(n-1)^2 - 4n(n-1)m + nM_1(G) - n(n-1)^2 + 4n(n-1)m - 4m^2} \\ &= \varepsilon(\overline{G}) + \sqrt{nM_1(G) - 4m^2}. \end{aligned}$$

If $\overline{G}$ is regular, then G is also regular. It is easy to check that $nM_1(G) = 4m^2$ and $QE(\overline{G}) = \varepsilon(\overline{G})$.

For graph energy, there is a Nordhaus-Gaddum-type result below.

Theorem 3 [9]. Let G be a graph with n vertices. Then

$$\varepsilon(G) + \varepsilon(\overline{G}) < \sqrt{2}n + (n-1)\sqrt{n-1}. \quad (5)$$

Similarly, we have

Theorem 4. If G and its complement $\overline{G}$ are connected graphs with order n, then

$$QE(G) + QE(\overline{G}) < \sqrt{2}n + (n-1)^{\frac{3}{2}} + 2\sqrt{nM_1(G) - 4m^2}. \quad (6)$$

Proof By (2) and (3), we have

$$QE(G) + QE(\overline{G}) \le \varepsilon(G) + \varepsilon(\overline{G}) + 2\sqrt{nM_1(G) - 4m^2}. \quad (7)$$

Put (5) into (7) and get

$$QE(G) + QE(\overline{G}) < \sqrt{2}n + (n-1)^{\frac{3}{2}} + 2\sqrt{nM_1(G) - 4m^2}.$$

Corollary 5. If G is a Q-borderenergetic graph of order n with m edges, satisfying G and $\overline{G}$ are connected graphs, then

$$QE(\overline{G}) < (\sqrt{2}-2)n + (n-1)^{\frac{3}{2}} + 2\sqrt{nM_1(G) - 4m^2} + 2.$$

Proof If G is a Q-borderenergetic graph of order n with m edges, by (6), then

$$2(n-1) + QE(\overline{G}) < \sqrt{2}n + (n-1)^{\frac{3}{2}} + 2\sqrt{nM_1(G) - 4m^2}.$$

Thus,

$$QE(\overline{G}) < (\sqrt{2}-2)n + (n-1)^{\frac{3}{2}} + 2\sqrt{nM_1(G) - 4m^2} + 2.$$

Theorem 6 [25]. If G is a Q-borderenergetic graph of order n with m edges, then

$$m > \frac{1}{4}\left(n - n^2 + \sqrt{n^2(n-1)^2 + 8nM_1(G)}\right). \tag{8}$$

Next a main result of this work is as follow.

Theorem 7. Suppose graphs G and its complement $\overline{G}$ are connected. If G is a Q-borderenergetic graph of order n with m edges, satisfying

$$\sqrt{M_1(G)} > \frac{n^{\frac{3}{2}}(n-1) + \sqrt{\varphi}}{2n},$$

where

$$\varphi = \sqrt{(\sqrt{2}-1)n^5 + (3-2\sqrt{2})n^4 - (13-5\sqrt{2})n^3 + (19-4\sqrt{2})n^2 - 9n}.$$

Then $\overline{G}$ is not a Q-borderenergetic graph.

Proof By contradiction. Suppose $\overline{G}$ is Q-borderenergetic. Set $M_1(G) = x$. By (6), we get

$$4(n-1) - \sqrt{2}n < (n-1)^{\frac{3}{2}} + 2\sqrt{nx - 4m^2}. \tag{9}$$

For a Q-borderenergetic graph, its order is greater than or equal to 6. Then we get $4(n-1) - \sqrt{2}n > 0$. Then square both sides of (9) and we have

$$\left(4(n-1) - \sqrt{2}n\right)^2 < \left((n-1)^{\frac{3}{2}} + 2\sqrt{nx - 4m^2}\right)^2,$$

and

$$\left(4(n-1) - \sqrt{2}n\right)^2 < \left(\sqrt{2}\sqrt{(n-1)^3 + 4nx - 16m^2}\right)^2.$$

So,

$$m^2 < \frac{4nx + n^3 + (4\sqrt{2}-12)n^2 + (19-4\sqrt{2})n - 9}{16}. \tag{10}$$

By Theorem 6, we have

$$m^2 > \frac{1}{16}\left(n - n^2 + \sqrt{n^2(n-1)^2 + 8nx}\right)^2,$$

and

$$m^2 > \frac{1}{16}\left[(n-n^2)^2 + n^2(n-1)^2 + 8nx + 2(n-n^2)\frac{\sqrt{2}}{2}\left(\sqrt{n^2(n-1)^2} + \sqrt{8nx}\right)\right],$$

$$= \frac{1}{16}\left[(2-\sqrt{2})n^2(n-1)^2 + 8nx - 4n(n-1)\sqrt{nx}\right].$$

Thus,

$$m^2 > \frac{8nx - 4n(n-1)\sqrt{nx} + (2-\sqrt{2})n^4 - (4-2\sqrt{2})n^3 + (2-\sqrt{2})n^2}{16}. \tag{11}$$

Let $\sqrt{x} = t$ and

$$f(t) = \frac{4nt^2 + n^3 + (4\sqrt{2} - 12)n^2 + (19 - 4\sqrt{2})n - 9}{16},$$

$$h(t) = \frac{8nt^2 - 4n^{\frac{3}{2}}(n-1)t + (2 - \sqrt{2})n^4 - (4 - 2\sqrt{2})n^3 + (2 - \sqrt{2})n^2}{16}.$$

Then

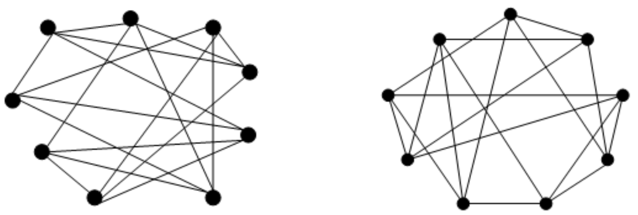
$$h(t) - f(t) = \frac{4nt^2 - 4n^{\frac{3}{2}}(n-1)t + (2 - \sqrt{2})n^4 - (5 - 2\sqrt{2})n^3 + (14 - 5\sqrt{2})n^2 - (19 - 4\sqrt{2})n + 9}{16} = \frac{H(t)}{16}.$$

As $\sqrt{M_1(G)} > \frac{n^{\frac{3}{2}}(n-1) + \sqrt{\varphi}}{2n}$, $t > \frac{n^{\frac{3}{2}}(n-1) + \sqrt{\varphi}}{2n}$, where

$$\varphi = \sqrt{(\sqrt{2} - 1)n^5 + (3 - 2\sqrt{2})n^4 - (13 - 5\sqrt{2})n^3 + (19 - 4\sqrt{2})n^2 - 9n}.$$

Then $H(t) > 0$, which is a contradiction.

Indeed, there are graphs satisfying the conditions given in theorem 7. Below we will give two examples to verify this fact by constructing graphs.



G_9^1 $\overline{G}_9^1$

Figure 1. *Two 4-regular graphs* G_9^1 *and* $\bar{G}_9^1$.

Example 1. The 4-regular graph $\overline{G}_9^1$ is the complement of G_9^1, see Figure 1. By direct computation, we have

$$QS_p(G_9^1)=\{2,2,2,2,5,5,5,5,8\},$$

$$QS_p(\overline{G}_9^1)=\{2,2,2,2,5,5,5,5,8\}.$$

It is easy to get that $\overline{G}_9^1$ is a Q-borderenergetic graph. But it satisfies

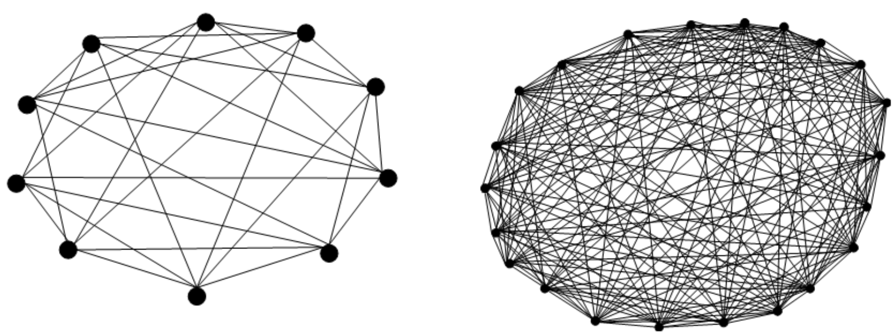
$$\sqrt{M_1(G_9^1)} = 12 < \frac{n^{\frac{3}{2}}(n-1) + \sqrt{\Delta}}{2n} \approx 20.3.$$

By Theorem 8 and Lemma 9, another example will be constructed by using the join of two graphs. The join of two graphs G_1 and G_2 is the graph $G_1 \nabla G_2$ in with the vertex set $V(G_1) \cup V(G_2)$ and the edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2.

Theorem 8 [6]. Let G be a k-regular Q-borderenergetic graph with n vertices. Then $G\nabla\overline{K}_{n-k}$ is Q-borderenergetic.

Lemma 9 [6]. Let $G^{(0)}$ be a k-regular Q-borderenergetic graph of order n with signless Laplacian eigenvalues $2k = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n$. Then for any $G^{(S)} \epsilon H(s \geq 1)$, then signless Laplacian spectrum of $G^{(S)}$ is the following:

$$Spes_Q(G^S) = \left(\mu_2 + s(n-k), \mu_3 + s(n-k), \cdots, \mu_n + s(n-k), \underbrace{n + (s-1)(n-k), \cdots, n + (s-1)(n-k)}_{s(n-k-1)}, n + (s-2)(n-k), 2n + 2(s-1)(n-k)\right). \quad (12)$$



G_{10} G_{22}

Figure 2. *Two regular Q-borderenergetic graphs* G_{10} *and* G_{22} *with* 10 *and* 22 *vertices, respectively.*

Example 2. The graph G_{10} is a 6-regular Q-borderenergetic graph. By using Theorem 8, repeatedly, a regular Q-borderenergetic graph G_{22} is obtained. Through direct calculation, the spectrum of $Q(G_{10})$ is as follow.

$$QS_p(G_{10}) = \{3,4,4,4,6,6,7,7,7,12\}.$$

From Lemma 9, we obtain the spectrum of $QS_p(G_{22})$ is

$$QS_p(G_{22}) = \left\{36,19,19,19,\underbrace{18,\cdots 18}_{11},16,16,16,15,14,14,14\right\}.$$

Obviously, the complement $\overline{G}_{22}$ of G_{22} is a 3-regular graph with order 22. Then the spectrum of $Q(\overline{G}_{22})$ is

$$QS_p(\overline{G}_{22}) = \left\{6,6,6,6,4,4,4,5,\underbrace{2,\cdots,2}_{11},1,1,1\right\}.$$

Thus, $QE(\overline{G}_{22}) = 34 \neq 42$ and $\overline{G}_{22}$ is not a Q-borderenergetic graph. It is easy to check that

$$\sqrt{M_1(G_{22})} \approx 84.43 > \frac{n^{\frac{3}{2}}(n-1)+\sqrt{\varphi}}{2n} \approx 82.33.$$

Then some lower bounds of the complement of a Q-borderenergetic graph are shown.

Theorem 10 [14]. Let G be a connected graph of order $n \geq 3$ with m edges, maximum degree Δ and minimum degree δ. Then

$$QE(G) \geq \Delta + \delta + \sqrt{(\Delta-\delta)^2 + 4\Delta} - \frac{4m}{n}, \tag{13}$$

with equality if and only if $G \cong K_{1,n-1}$.

Corollary 11. Let G be a connected graph of order $n \geq 3$ with m edges, maximum degree Δ and minimum degree δ. Then

$$QE(G) + QE(\overline{G}) \geq 2\sqrt{(\Delta-\delta)^2 + 2[n-1-(\Delta+\delta)]}.$$

Proof According to Theorem 6, if $\overline{G}$ is the complement of G, then $\overline{G}$ has the maximum degree $n-1-\delta$ and the minimum degree $n-1-\Delta$. Let $\overline{m}$ be the size of $\overline{G}$. Then

$$QE(\overline{G}) \geq (n-1-\delta) + (n-1-\Delta) + \sqrt{[(n-1-\delta)-(n-1-\Delta)]^2 + 4(n-1-\delta)} - \frac{4\overline{m}}{n},$$

$$= 2(n-1) - (\Delta+\delta) + \sqrt{(\Delta-\delta)^2 + 4(n-1-\delta)} - \frac{4\overline{m}}{n}.$$

Then we get

$$QE(G) + QE(\overline{G}) \geq \Delta + \delta + \sqrt{(\Delta-\delta)^2 + 4\Delta} - \frac{4m}{n} + 2(n-1) - (\Delta+\delta) + \sqrt{(\Delta-\delta)^2 + 4(n-1-\delta)} - \frac{4\overline{m}}{n},$$

$$\geq 2(n-1) + \sqrt{(\Delta-\delta)^2 + 4\Delta} + \sqrt{(\Delta-\delta)^2 + 4(n-1-\delta)} - \frac{4(m+\overline{m})}{n},$$

$$\geq \sqrt{(\Delta-\delta)^2 + 4\Delta} + \sqrt{(\Delta-\delta)^2 + 4(n-1-\delta)}$$

$$\geq 2\sqrt{(\Delta-\delta)^2 + 2[n-1-(\Delta+\delta)]}.$$

Corollary 12. Let G be a connected graph of order $n \geq 3$ with m edges, maximum degree Δ and minimum degree δ. If G is a *Q-borderenergetic graph,* then

$$QE(\overline{G}) \geq 2\sqrt{(\Delta-\delta)^2 + 2[n-1-(\Delta+\delta)]} - 2(n-1).$$

3. Conclusion

In this paper, we mainly survey the signless Laplacian energy of the complement of a Q-borderenergetic graph. Theory 7, as a main result, can further be improved if the condition on the parameter $M_1(G)$ is ignored. In addition, the graphs attained the lower bounds in Corollary 12 can be considered by the properties of complements of graphs.

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