

The Application of Eigenvectors for the Construction of Minimum-Energy Wavelet Frames Based on FMRA

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Abstract: In 1974, J. Morlet raised the concept of wavelet transform and established the inversion formula through the experience of physical intuition and signal processing. In 1986, Y. Meryer created a real small wave base, and the wavelet analysis began to flourish after a multi scale analysis of the same method of constructing the small wave base with S. Mallat. In order to analyze and deal with non-stationary signals, a series of new signal analysis theories are proposed.: Short Time Fourier Transform, time-frequency analysis, wavelet transform, and fractional Fourier transform and so on. In this paper, an explicit algorithm is given to construct the minimum-energy frames based on frame multiresolution analysis via characteristic vectors of the mask matrix. In section 2, we show the structure of minimum-energy wavelet frames in terms of their masks (Lemma 1) and discuss that we should eliminate the correlation of the rows of the mask matrix by the polyphase decomposition technique. Based on FMRA, an explicit algorithm is given to construct this frames. By this method, all the minimum-energy wavelet frames can be obtained. As an application, several examples are showed to explain this method in section 3. This method can also be applied in other fields of wavelet analysis.

Keywords: Frame Multiresolution Analysis, Polyphase Decomposition, Minimum-Energy Frames

1. Introduction

In 1974, an French engineer, J. Morlet, raised the concept of wavelet transform and established the inversion formula through the experience of physical intuition and signal processing. In 1986, Y. Meryer, created a real small wave base, and the wavelet analysis began to flourish after a multi scale analysis of the same method of constructing the small wave base with S. Mallat. In order to analyze and deal with non-stationary signals, a series of new signal analysis theories are proposed and developed: Short Time Fourier Transform, time-frequency analysis, wavelet transform, and fractional Fourier transform and so on.

This paper deals with the study of compactly supported minimum-energy wavelet frames corresponding to a single refinable function with compact support. It is well known that the multiresolution analysis (MRA for short) is a systematic method to construct orthonormal wavelet bases for $L^p(\mathbb{R})$ [1-8].

MRA requires that the refinement mask $H(\omega)$ should satisfy $|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1$ [9, p132]. However, there exist many refinable functions whose mask $H(\omega)$ possesses $|H(\omega)|^2 + |H(\omega + \pi)|^2 \leq 1$. So we want to know whether these refinable functions can generate tight wavelet frames, especially minimum-energy wavelet frames. In 1998, J. J. Benedetto and S. Li [10] introduced the theory of frame multiresolution analysis (FMRA for short). FMRA is an extension of the concept of MRA. Based on FMRA, we can construct tight wavelet frames associated with a given refinable function.

Definition 1 (FMRA) A FMRA associated with a dilation factor $M \in \mathbb{R}, M \geq 2$ is a sequence of close subspaces of $L^2(\mathbb{R})$ satisfying the following conditions:

1. $\{V_j\} \subset \{V_{j+1}\}, j \in \mathbb{Z}$;
2. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$

3. $f(x) \in V_j$ if and only if $f(Mx) \in V_{j+1}$;

4. There exists a function $\varphi(x) \in V_0$ such that $\{\varphi(x - n) : n \in Z\}$ forms a frame in V_0 .

The function $\varphi(x)$ is called a frame refinable function for the FMRA. It is emphasized that the shifts of $\varphi(x)$ form a frame, not necessarily an orthonormal or a Riesz base of V_0 as MRA. In this paper, we fix $M=2$. In this case, Charles K. Chui and Wenjie He [11] discussed and constructed minimum-energy frames by the unitary matrix extension [see 12]. When $M=3$, Cui Lihong, Cheng Zhengxing and Yang Shouzhi gave a sufficient and necessary condition to the tight wavelet frames in [13]. Motivated by the work of [11] and [13], we construct the minimum-energy wavelet frames via characteristic vectors of the mask matrix.

Definition 2 (Minimum-energy wavelet frames [11]) Let $\varphi(x) \in L^2(\mathbb{R})$ with $\hat{\varphi} \in L^\infty$, $\hat{\varphi}$ continuous at 0, and $\hat{\varphi}(0) = 1$ be a scaling function that generates a FMRA. Then a finite family of functions $\psi = \{\psi^1, \psi^2, \dots, \psi^N\} \subset V_1$ is called a minimum-energy wavelet frame associated with $\varphi(x)$, if

$$\sum_{k \in Z} |\langle f, \varphi_{1,k} \rangle|^2 = \sum_{k \in Z} |\langle f, \varphi_{0,k} \rangle|^2 + \sum_{i=1}^N \sum_{k \in Z} |\langle f, \psi_{0,k}^i \rangle|^2,$$

All $f \in L^2(\mathbb{R})$.

The minimum-energy wavelet frame ψ is necessarily a tight wavelet frame in $L^2(\mathbb{R})$ with frame bound equals to 1. One of the advantage of this frame is that it can avoid the complication of the change of bases using the same wavelets as the orthonormal bases [see 11]. In this paper, an explicit algorithm is given to construct this frames. The paper is organized as follows. In section 2, we show the structure of minimum-energy wavelet frames in terms of their masks (Lemma 1) and discuss that we should eliminate the correlation of the rows of the mask matrix by the polyphase decomposition technique. The explicit algorithm is also given in section 2. The last section is devoted to some examples obtained by this algorithm.

2. Preliminaries and Main Results

2.1. Preliminaries

Let $\{V_j\}_{j \in Z}$ generates an FMRA in $L^2(\mathbb{R})$ and $\psi = \{\psi^1, \psi^2, \dots, \psi^N\} \subset V_1$.

Since $V_0 \subset V_1 = \overline{\text{Span}\{\varphi(2x - n) : n \in Z\}}$, we have $N+1$ sequences $\{h_n\}_{n \in Z} \in \ell^2$ and $\{g_n^1\}_{n \in Z} \in \ell^2$ such that

$$\begin{cases} \varphi(x) = \sum_{n \in Z} h_n \varphi(2x - n) \\ \psi^1(x) = \sum_{n \in Z} g_n^1 \varphi(2x - n) \end{cases} \quad (1)$$

here and throughout, $l = 1, 2, \dots, N$.

In this paper, the Fourier transform of an integrable function $f(x)$ is defined as $\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-i\omega x} dx$. Taking Fourier transform at both sides of (1) leads to

$$\begin{cases} \hat{\varphi}(x) = \frac{1}{2} \sum_{n \in Z} h_n e^{-\frac{i}{2}\omega n} \hat{\varphi}\left(\frac{\omega}{2}\right) \\ \hat{\psi}^1(x) = \frac{1}{2} \sum_{n \in Z} g_n^1 e^{-\frac{i}{2}\omega n} \hat{\varphi}\left(\frac{\omega}{2}\right) \end{cases} \quad (2)$$

Set $H(\omega) = \frac{1}{2} \sum_{n \in Z} h_n e^{-i\omega n}$

And $G^1(\omega) = \frac{1}{2} \sum_{n \in Z} g_n^1 e^{-i\omega n}$, then (2) is equivalent to

$$\begin{cases} \hat{\varphi}(\omega) = H\left(\frac{\omega}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right) \\ \hat{\psi}^1(\omega) = G^1\left(\frac{\omega}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right) \end{cases} \quad (3)$$

The 2π -periodic functions $H(\omega)$ and $G^1(\omega)$ are called the refinement and the wavelet masks respectively. With $H(\omega)$ and $G^1(\omega)$, a $2 \times (N + 1)$ matrix can be formulated as

$$M(\omega) = \begin{pmatrix} H(\omega) & G^1(\omega) & \dots & G^N(\omega) \\ H(\omega + \pi) & G^1(\omega + \pi) & \dots & G^N(\omega + \pi) \end{pmatrix} \quad (4)$$

Charles K. Chui and Wenjie [11] gave the structure of the minimum-energy wavelet frame as follows.

Lemma 1 Let $\varphi(x) \in L^2(\mathbb{R})$ with $\hat{\varphi} \in L^\infty$, $\hat{\varphi}$ continuous at 0, and be a refinable function that generates a FMRA in the sense of Definition 1. And Let $H(\omega)$ and $G^1(\omega)$ be the masks concerning with φ and $\psi = \{\psi^1, \psi^2, \dots, \psi^N\}$. Then ψ is a minimum-energy wavelet frame if and only if

$$M(\omega)M^*(\omega) = I_2, \quad \text{a.e.}\omega \quad (5)$$

here $M^*(\omega)$ represents the complex conjugate of the transpose of $M(\omega)$.

It should be emphasized that Charles K. Chui and Wenjie [11] have pointed out that the refinable function $\varphi(x)$ generates a FMRA tight wavelet frame if and only if the refinable mask $H(\omega)$ satisfy $|H(\omega)|^2 + |H(\omega + \pi)|^2 \leq 1$. From Lemma 1, the construction of wavelet frame can be reduced to

the problem of extending a vector matrix $(H(\omega), H(\omega + \pi))^*$ to a unitary matrix as (4). That is, we need to seek N functions $G^1(\omega), G^2(\omega), \dots, G^N(\omega)$ such that (5) is satisfied. Note the rows of (4) are correlative, so we should remove this feature by using the polyphase decomposition technique [see 13, p106, also see 9, p318] first. Similarly to [13] and for completeness, this technique is introduced briefly. $H(\omega)$ and $G^1(\omega)$ can be written in their polyphase forms respectively as

$$\begin{cases} H(\omega) = \frac{1}{\sqrt{2}} (H_1(\omega) + e^{-i\omega}H_2(\omega)) \\ G^1(\omega) = \frac{1}{\sqrt{2}} (G^{11}(\omega) + e^{-i\omega}G^{12}(\omega)) \end{cases} \quad (6)$$

Write

$$N(\omega) = \begin{pmatrix} H_1(\omega) & G^{11}(\omega) & \dots & G^{N1}(\omega) \\ H_2(\omega + \pi) & G^{12}(\omega + \pi) & \dots & G^{N2}(\omega + \pi) \end{pmatrix} \quad (7)$$

Since $H_1(\omega), H_2(\omega)$ and $G^{11}(\omega), G^{12}(\omega)$ are π -periodic functions, we have

$$M(\omega) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\omega} \\ 1 & -e^{-i\omega} \end{pmatrix} N(\omega) \quad (8)$$

Then (5) means

$$N(\omega)N^*(\omega) = I_2, \quad \text{a.e.}\omega \quad (9)$$

Remark 1 The matrix $N(\omega)$ is the polyphase decomposition of $M(\omega)$ and (9) and (5) look very alike. We can extensive $N(\omega)$ to an unitary matrix to obtain a minimum-energy wavelet frame. The difference of (9) and (5) is that the rows of $N(\omega)$ are not correlative. In some applications, we hope the refinable functions and frames have some special properties such as symmetric or anti-symmetric. Since this algorithm needs annihilate the correlative of $M(\omega)$ and the polyphase decompositions do not keep the symmetric or anti-symmetric feature, we should return to (4) and (5) to obtain a symmetric or anti-symmetric frame. Charles K. Chui and Wenjie [11] gave an ingenious constructive method.

2.2. Main Results

In this subsection, we prove that if the refinement mask $H(\omega)$ satisfies $|H(\omega)|^2 + |H(\omega + \pi)|^2 \leq 1$. Then there exists an explicit algorithm to construct a minimum-energy wavelet frame. This method is motivated by Cui Lihong, Cheng Zhengxing and Yang Shouzhi [12].

Since $|H_1(\omega)|^2 + |H_2(\omega)|^2 = |H(\omega)|^2 + |H(\omega + \pi)|^2$, we have $|H_1(\omega)|^2 + |H_2(\omega)|^2 \leq 1$. Set

$$G(\omega) = \begin{pmatrix} G^{11}(\omega) & G^{21}(\omega) & \dots & G^{N1}(\omega) \\ G^{12}(\omega) & G^{22}(\omega) & \dots & G^{N2}(\omega) \end{pmatrix} \quad (10)$$

then (9) is reformulated as

$$\begin{pmatrix} H_1(\omega) \\ H_2(\omega) \end{pmatrix} \begin{pmatrix} \overline{H_1(\omega)} & \overline{H_2(\omega)} \end{pmatrix} + G(\omega)G^*(\omega) = I_2 \quad (11)$$

or equivalently,

$$\begin{aligned} G(\omega)G^*(\omega) &= I_2 - \begin{pmatrix} H_1(\omega) \\ H_2(\omega) \end{pmatrix} \begin{pmatrix} \overline{H_1(\omega)} & \overline{H_2(\omega)} \end{pmatrix} \\ &= \begin{pmatrix} 1 - |H_1(\omega)|^2 & -H_1(\omega)\overline{H_2(\omega)} \\ -\overline{H_1(\omega)}H_2(\omega) & 1 - |H_2(\omega)|^2 \end{pmatrix} \end{aligned} \quad (12)$$

By simple calculation, the characteristic roots of $G(\omega)G^*(\omega)$ are

$$\lambda_1 = 1, \quad \lambda_2 = 1 - |H_1(\omega)|^2 - |H_2(\omega)|^2, \quad (13)$$

and the corresponding unit characteristic vectors are

$$\begin{cases} \alpha = \left(-\overline{H_2(\omega)}, \overline{H_1(\omega)} \right)^T / \Delta \\ \beta = \left(H_1(\omega), H_2(\omega) \right)^T / \Delta \end{cases} \quad (14)$$

$$\Delta^2 = |H_1(\omega)|^2 + |H_2(\omega)|^2. \quad (15)$$

Since $|H_1(\omega)|^2 + |H_2(\omega)|^2 \leq 1$, by Riesz Lemma [9, Lemma 6.1.3], there exists a polynomial $H_3(\omega)$ such that

$$|H_3(\omega)|^2 = 1 - |H_1(\omega)|^2 - |H_2(\omega)|^2. \quad (16)$$

Note $G(\omega)G^*(\omega)$ is a complex symmetrical matrix, $G(\omega)G^*(\omega)$ can be written as

$$\begin{aligned} G(\omega)G^*(\omega) &= (\alpha, \beta) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\alpha, \beta)^* \\ &= (\alpha, \beta) \begin{pmatrix} 1 & 0 \\ 0 & 1 - |H_1(\omega)|^2 - |H_2(\omega)|^2 \end{pmatrix} (\alpha, \beta)^* \\ &= (\alpha, \beta) \begin{pmatrix} 1 & 0 \\ 0 & |H_3(\omega)|^2 \end{pmatrix} (\alpha, \beta)^* \\ &= ((\alpha, \beta) \begin{pmatrix} 1 & 0 \\ 0 & |H_3(\omega)| \end{pmatrix} g(\omega)) ((\alpha, \beta) \begin{pmatrix} 1 & 0 \\ 0 & |H_3(\omega)| \end{pmatrix} g(\omega))^*, \end{aligned} \quad (17)$$

here and throughout, $g(\omega)$ is a $2 \times N$ matrix and satisfies $g(\omega) \cdot g(\omega)^* = I_2$. Therefore (17) means that

$$G(\omega) = (\alpha, \beta) \begin{pmatrix} 1 & 0 \\ 0 & H_3(\omega) \end{pmatrix} g(\omega). \quad (18)$$

All the above leads to the following Theorem 1.

Theorem 1 Let $\varphi(x) \in L^2(\mathbb{R})$ with $\hat{\varphi} \in L^\infty$, $\hat{\varphi}$ continuous at 0, and $\hat{\varphi}(0) = 1$, be a refinable function that generates a FMRA whose mask $H(\omega)$ satisfies

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 \leq 1, \quad \text{a.e.}\omega \quad (19)$$

and let $H_1(\omega)$ and $H_2(\omega)$ be the polyphase components of $H(\omega)$ respectively. Then there exists a minimum-energy wavelet frame $\psi = \{\psi^1, \psi^2, \dots, \psi^N\} \subset V_1$ associated with $\varphi(x)$. Furthermore, all the minimum-energy wavelet frames can be written in the sense of their masks as

$$G(\omega) = \frac{1}{\Delta} \begin{pmatrix} -\overline{H_2(\omega)} & H_1(\omega) \\ H_1(\omega) & H_2(\omega) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H_3(\omega) \end{pmatrix} g(\omega), \quad (20)$$

where $\Delta^2 = |H_1(\omega)|^2 + |H_2(\omega)|^2$ and $H_3(\omega)$ satisfies

$$|H_1(\omega)|^2 + |H_2(\omega)|^2 + |H_3(\omega)|^2 = 1. \quad (21)$$

Remark 2 If the refinable function $\varphi(x)$ generates an orthonormal wavelet base in $L^2(\mathbb{R})$, then the FMRA is a standard MRA and the refinement mask $H(\omega)$ satisfies

$|H_1(\omega)|^2 + |H_2(\omega)|^2 = 1$. In this case, $H_3(\omega) = 0$. Since an orthonormal wavelet base also is a minimum-energy wavelet frame, Theorem 1 includes this case in which the corresponding orthonormal wavelet base is $(-\overline{H_2(\omega)}, H_1(\omega))^T / \Delta$. Examples 1 explains this case.

Remark 3 By Theorem 1, if we choose a different $g(\omega)$, we can find all the compact support minimum-energy frames consists of N wavelet $\psi^1, \psi^2, \dots, \psi^N$ associated with a given compactly support refinable function $\varphi(x)$. When $N=2$, the minimum-energy wavelet frame $\psi = \{\psi^1, \psi^2\}$ is obtained as [9].

3. Examples

In this section, several examples associated with the cardinal B-splines are given. It is well known that in the development of wavelet analysis, cardinal B-splines serve as a canonical example of scaling functions that generate MRA in $L^2(\mathbb{R})$. The m order cardinal B-splines $N_m(x)$, $m \geq 2$ is defined inducting by

$$N_m = \int_0^1 N_{m-1}(t-x) dx \quad (22)$$

with $N_1(x)$ denoting the characteristic function of the unit interval $[0, 1]$ (see [14, p188]). The mask of $N_m(x)$ is

$$H^m(\omega) = \left(\frac{1 + e^{-i\omega}}{2} \right)^m. \quad (23)$$

And we can see that $H^m(\omega)$ satisfies

$$|H^m(\omega)|^2 + |H^m(\omega + \pi)|^2 \leq |H^1(\omega)|^2 + |H^1(\omega + \pi)|^2 = 1. \quad (24)$$

Example 1 (Haar wavelet) This is the special case when $m=1$ and known as the Haar wavelet. The Haar function is $N_1(x)$ and the refinement mask $H^1(\omega) = \frac{1 + e^{-i\omega}}{2}$ satisfies $|H^1(\omega)|^2 + |H^1(\omega + \pi)|^2 = 1$. It is easy to find the minimum-energy wavelet frame mask is

$$G(\omega) = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} g(\omega). \quad (25)$$

Example 2 (Linear B-splines) When $m=2$,

$$H^2(\omega) = \left(\frac{1 + e^{-i\omega}}{2} \right)^2 = \frac{1}{4} (1 + 2e^{-i\omega} + e^{-2i\omega}). \quad (26)$$

The polyphase decompositions are obtained as follows

$$H_1^2(\omega) = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} e^{-2i\omega} \quad (27)$$

And

$$H_2^2(\omega) = \frac{\sqrt{2}}{2}. \quad (28)$$

there exists $H_3^2(\omega) = \frac{\sqrt{2}}{4} (1 - e^{-2i\omega})$ such that

$|H_1(\omega)|^2 + |H_2(\omega)|^2 + |H_3(\omega)|^2 = 1$. By Theorem 1, all the minimum-energy wavelet frame masks associated with $N_2(x)$ are

$$G(\omega) = \frac{1}{\Delta} \begin{pmatrix} -\overline{H_2(\omega)} & H_1(\omega) \\ H_1(\omega) & H_2(\omega) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H_3(\omega) \end{pmatrix} g(\omega), \quad (29)$$

Here

$$\Delta = \frac{1}{8} (6 + e^{-2i\omega} + e^{2i\omega}). \quad (30)$$

In this case, if we choose $g(\omega)$ as

$$g(\omega) = \frac{1}{\Delta} \begin{pmatrix} \frac{1}{2}(1 + e^{-2i\omega}) & -\frac{\sqrt{2}}{4}(1 - e^{-2i\omega}) \\ \frac{\sqrt{2}}{4}(1 - e^{2i\omega}) & \frac{1}{2}(1 + e^{2i\omega}) \end{pmatrix}, \quad (31)$$

It is easy to find a minimum-energy frame in terms of masks

$$Q_1(\omega) = -\frac{1}{4} + \frac{1}{2}e^{2i\omega} - \frac{1}{4}e^{-2i\omega}, \quad (32)$$

and

$$Q_2(\omega) = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}e^{-2i\omega}. \quad (33)$$

Furthermore, it is easy to see the corresponding wavelet function $\psi^1(x)$ is symmetric and $\psi^2(x)$ is anti-symmetric. This result was also given in [11].

Example 3 (Quadratic B-splines) When $m=3$,

$$H^3(\omega) = \frac{1}{8}(1 + 3e^{-i\omega} + 3e^{-2i\omega} + e^{-3i\omega}). \quad (34)$$

Similarly to Example 2, we obtain

$$H_1^3(\omega) = \frac{\sqrt{2}}{8}(1 + 3e^{-2i\omega}), \quad (35)$$

$$H_2^3(\omega) = \frac{\sqrt{2}}{8}(3 + e^{-2i\omega}), \quad (36)$$

and

$$H_3^3(\omega) = \frac{\sqrt{3}}{4}(1 - e^{-2i\omega}). \quad (37)$$

So all the corresponding minimum-energy wavelet tight frame masks are

$$G(\omega) = \frac{1}{\Delta} \begin{pmatrix} -\overline{H_2^3(\omega)} & H_1^3(\omega) \\ \overline{H_1^3(\omega)} & H_2^3(\omega) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H_3^3(\omega) \end{pmatrix} g(\omega), \quad (38)$$

here

$$\Delta = \frac{1}{16}(10 + 3e^{2i\omega} + 3e^{-2i\omega}). \quad (39)$$

Now we choose

$$g(\omega) = \frac{1}{\Delta} \begin{pmatrix} \frac{\sqrt{3}}{4}(1 + e^{-2i\omega}) & \frac{1}{2}e^{-2i\omega} \\ -\frac{1}{2}e^{2i\omega} & \frac{\sqrt{3}}{4}(1 + e^{2i\omega}) \end{pmatrix}, \quad (40)$$

then an anti-symmetric minimum-energy frame is obtained as

$$Q_1(\omega) = -\frac{\sqrt{3}}{4}(1 - e^{-i\omega}) \quad (41)$$

and

$$Q_2(\omega) = \frac{1}{8}(1 + 3e^{-i\omega} - 3e^{-2i\omega} - e^{-3i\omega}). \quad (42)$$

4. Conclusion

This paper deals with the study of compactly supported minimum-energy wavelet frames corresponding to a single refinable function with compact support. In section 2, we show the structure of minimum-energy wavelet frames in terms of their masks (Lemma 1) and discuss that we should eliminate the correlation of the rows of the mask matrix by the polyphase decomposition technique. Based on FMRA, an explicit algorithm is given to construct these frames. In section 3, some examples are given by this algorithm. This method can also be applied in other fields of wavelet analysis.

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