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A Robust Preconditioned Iterative Method for the Navier-Stokes Equations with High Reynolds Numbers

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Abstract: In this paper, we proposed a new solver for the Navier-Stokes equations coming from the channel flow with high Reynolds number. We use the preconditioned Krylov subspace iterative methods such as Generalized Minimum Residual Methods (GMRES). We consider the variation of the Hermitian and Skew-Hermitian splitting to construct the preconditioner. Convergence of the preconditioned iteration is analyzed. We can show that the proposed preconditioner has a robust behavior for the Navier-Stokes problems in variety of models. Numerical experiments show the robustness and efficiency of the preconditioned GMRES for the Navier-Stokes problems with Reynolds numbers up to ten thousands.

Keywords: Preconditioning, GMRES, Navier-Stokes, High Reynolds Number, Iterative Methods

1. Introduction

High Reynolds-number incompressible flow has been investigated widely recently years. Turbulent flows with Reynolds number of the order of 10^3 and greater are of interest because this is the range of Reynolds numbers relevant to many industrial applications ([7-8]). The aim of this paper is to study the governing equations: unsteady and steady Navier-Stokes equations with high Reynolds numbers. We consider the incompressible viscous fluid problems with the following form:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \text{ in } \Omega \times [0, \Gamma] \quad (1)$$

$$\nabla \cdot u = 0 \text{ in } \Omega \times [0, \Gamma] \quad (2)$$

$$\mathfrak{B}u = g \text{ in } \partial\Omega \times [0, \Gamma] \quad (3)$$

$$u(\mathrm{x}, 0) = u_0 \text{ in } \Omega \quad (4)$$

Equations (1) to (4) are also known as the Navier-Stokes equations. Here is an open set of, where d = 2, or d =3, with boundary Ω ; the variable $u = u(\mathrm{x}, t) \in \mathbb{R}^{\mathrm{d}}$ is a vector-valued function representing the velocity of the fluid, and the scalar function $p = p(x, t) \in \mathbb{R}$ represents the pressure. The source function f is given on Ω. Here $\nu > 0$ is a given constant called the kinematic viscosity, which is $\nu = O(Re^{-1})$. Re is the Reynolds number: $Re = \frac{VL}{\backslash nu}$, where V denotes the mean velocity and L is the diameter of Ω (see [4]). Also, $\Delta$is the (vector) Laplacian operator in d dimensions, ∇ is the gradient operator, and ∇ is the divergence operator. In (3) $\mathfrak{B}$ is some boundary operator; for example, the Dirichlet boundary condition $u = g$ or Neumann boundary condition $\frac{\partial u}{\partial n} = g$, where n denotes the outward-pointing normal to the boundary; or a mixture of the two. Different types of flows may have different types of boundary conditions. If a flow is inside a container, then there must be no flow across the boundary. In this case, we have the boundary condition $u \cdot n = 0$ on $\partial\Omega$. The fundamental principles used to establish these partial differential equations (PDEs) are conservation of mass and conservation of momentum. Equation (1) represents the conservation of momentum, and it is called the convection form of the momentum equation. Equation (2) represents the conservation of mass, since for an incompressible and homogeneous fluid the density is constant both with respect to time and the spatial coordinates. Equations (1) – (4) describe the dynamic behavior of Newtonian fluids, such as water, oil and other liquids.

We use fully implicit time discretization and Picard linearization to obtain a sequence of Oseen problems, i.e. linear problems of the form

$$\alpha u - v\Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in} \quad \Omega \tag{5}$$

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega \tag{6}$$

$$\mathfrak{B}u = g \quad \text{on} \quad \partial\Omega \tag{7}$$

Where $\alpha > 0$, with $\alpha = O(\frac{1}{\delta t})$. Here δt denotes the time step. Vector v is a known vector which is obtained from the previous Picard iteration. For an example $v = u_{k-1}$. Equations (5) -- (7) are referred to as the generalized Oseen problem. If $\alpha = 0$, we have the steady-state problem. If $v = 0$, we have the Stokes problem.

The discretization of equations (1)-(4) will result a linear system

$$\mathrm{A}x = b,$$

where A writes as $\begin{pmatrix} F & & B_1^T \\ & F & B_2^T \\ -B_1 & -B_2 & C \end{pmatrix}$.

Here $F = \alpha I - vH + S$, where H is the discretization of Laplacian operator in two dimensions, and I is the identity matrix. S is the discretization matrix of the convection term $(u \cdot \nabla)$, the rectangular matrix B_i^T (i = 1, 2) represents the discrete gradient operator while B_i represents its adjoint, the (negative) divergence operator. In order to make the matrix A invertible, we add a stabilization term $C = h^2 I$, where $h = \frac{1}{n}$ with n the grid size of the discretization. If D is a zero matrix, then we obtain the coefficient matrix of the Stokes equations. α is the $O(\frac{1}{\Delta t})$, where Δt is the time step.

An efficient solver for the Navier-Stokes problems can be realized by combining the good choice of preconditioners and Krylov subspace iterative methods. Details of preconditioning and iterative solvers can be found in many textbooks (see [1] and [2] [4]). Most existing methods work well for small Reynolds numbers but fail as the Reynolds number increases to the range $10^3 - 10^5$. In this paper, we propose a preconditioner with Hermitian and Skew-Hermitian splitting. The new preconditioned iterative solvers provide a robust behavior under the high Reynolds numbers (or small viscosity).

2. Numerical Solvers

Our aim is to study the numerical solver of the Navier-Stokes problems with high Reynolds numbers. We consider the Picard iteration to linearize the nonlinear problem (1) – (4). This corresponds to a simple fixed point iteration strategy. Although the rate of of convergence of Picard iteration is linear in general, its radius of convergence is bigger than Newton iteration.

At each Picard iteration, a numerical solver for the linear system below is required.

$$\begin{pmatrix} F & B^T \\ B & C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{8}$$

Where the structure of F is the discetization of the term $\alpha - v\Delta + (u \cdot \nabla)$. For Picard iteration, F is a block diagonal matrix with each of the diagonal block is a discrete convection-diffusion operator. The matrix C is the stabilization term. If we use Marker-and-Cell (MAC) discretization, C is a zero matrix.

The iterative method we used in this paper is the generalized minimum residual method (GMRES). For nonsymmetric problems, this method represents the standard approach for constructing iterates satisfying an optimality condition. We implemented GMRES with the proposed preconditioner to speed up the convergence of GMRES. In fact, we could also use other Krylov subspace iterative methods such as BICGSTAB. (see [4]).

Krylov subspace iterative methods are tend to be low cost and faster convergence compared with the direct method such as Gaussian Eliminations. However, all Krylov subspace methods need accelerate convergence by applying *preconditioning.* Preconditioning is a key ingredient for the success of Krylov subspace methods. Preconditioning is a transformation of the original system into another system such that the new system has more favorable properties for iterative solutions. A *preconditioner* P is a matrix that effects such transformation. After we apply the preconditioner matrix P to the original matrix A, the preconditioned system P^{-1}A is supposed to have a better spectral properties. If the matrix is symmetric, the rate of convergence of the Conjugate Gradient (CG) method or Minimum Residual Method (MINRES) depend on the distribution of the eigenvalues of the matrix A. If the preconditioned matrix P ^{1}A has a smaller spectral condition number or the eigenvalues are clustered around 1, then we can expect a fast rate of convergence. For nonsymmetric (nonnormal) problems the situation is more complicated and the eigenvalues may not describe the convergence of nonsymmetric matrix iterations like General Minimum Residual Method (GMRES); see the discussion in [4]. Nevertheless, a clustered spectrum (away from 0) often results in rapid convergence, especially if the departure from normality of the preconditioned matrix is not too high.

The Hermitian and Skew-Hermitian splitting (HSS) preconditioner is based on Herimitian and skew-Hermitian splitting of the coefficient matrix. Letting

$$H = \frac{1}{2}(A + A^T), K = \frac{1}{2}(A - A^T),$$

we have the following splitting of A into its symmetric and skew-symmetric parts:

$$A = \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & C \end{pmatrix} + \begin{pmatrix} K & B^T \\ -B & 0 \end{pmatrix} = H + K$$

Note that H, the symmetric part of A, is symmetric positive semidefinite since H and C are. K is a skew symmetric matrix. Let ρ>0 be a parameter, the HSS preconditioner is defined as follows:

$$P_{hss} = \frac{1}{2\rho}\,(H + \rho I_{m+n})(K + \rho I_{m+n})$$

where, I_{m+n} is the identity matrix of size m+n. To Solve this preconditioner, it requires solving a shifted Hermitian system and a shifted Skew Hermitian system. This preconditioner was first proposed by Benzi and Golub. Then it is used as a preconditioner for the Oseen problem in rotation form in [3]. HSS preconditioner works better on the Navier-Stokes problems with small viscosity or high Reynolds number. Before we apply the HSS preconditioner, we modify the preconditioner in the following format:

$$\mathrm{H} = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},\ \mathrm{K} = \begin{pmatrix} \alpha\mathrm{I} + \mathrm{K} & \mathrm{B}^{\mathrm{T}} \\ \mathrm{B} & \mathrm{C} \end{pmatrix},$$

Where M is the discretizaion of $-\nu\Delta$, K is the discretization of $u \cdot \nabla$. Again in order to make both matrices invertible, we shift H and S.

$$P_m = \frac{1}{2\rho}\,(H + \rho I_{m+n})(K + \rho I_{m+n}).$$

The shifting parameter plays an important role here. It is quite challenging to identify the optimal choice of ρ since it is case dependent.

To solve this preconditioner, we need to solve two individual systems:

$$\mathrm{Hv} = \mathrm{w};\ \mathrm{Sy} = \mathrm{v};$$

The shifted H is a shifted Laplacian. It is quite easy to solve. H can be solved by Conjugate Gradient (CG) with incomplete Cholesky preconditioner or multigrid method. The system with the coefficient matrix S is the main cost of applying the preconditioning. GMRES with incomplete LU preconditioning is considered. In both cases, the inner iterations contains incomplete Cholesky or incomplete LU factorizations. For the inexact inner iteration discussions, see [10].

3. Numerical Experiments

In this section, we show the numerical experimental results for the Navier-Stokes problems with Reynolds numbers varies from $10^3 - 10^5$. All results were computed in MATLAB 7.1.0 on one processor of an AMD Opteron with 32 GB of memory.

In all experiments, symmetric diagonal scaling was applied before forming the preconditioners. We found that this scaling is not only beneficial to convergence, but also it makes finding (nearly) optimal values of the shift ρ easier. Of course, the right-hand side and the solution vector were scaled accordingly. We used right preconditioning in all cases. The outer iteration (full GMRES) was stopped when $\frac{\|r_k\|_2}{\|r_0\|_2} < 10^{-6}$, where r_k denotes the residual vector at step k. For the results presented in this section, the symmetric positive definite systems were solved 'exactly' by means of the sparse Cholesky factorization available in MATLAB, in combination with an approximate minimum degree ordering to reduce fill-in. For the sparse, nonsymmetric Schur complement system, we used the sparse LU solver available in MATLAB with the original (lexicographic) ordering. We found this to be faster than a minimum degree ordering, probably because the need for pivoting makes the fill-reducing ordering ineffective or even harmful.

3.1. The Unsteady Oseen Problem Using MAC Discretizations

In this experiment, we tested the classic unsteady Oseen problem on a unit square domain (5) – (7). MAC discretization is used. In this case, no stabilization term is needed. Figure 1 shows the staggered grid using MAC discretizations.

Table 1, 2 and Table 3 are the experimental results for the iteration counts of preconditioned GMRES for the unsteady Oseen problem with viscosity 0.001 to 10^{-5}. We can see that the modified HSS preconditioner works even better for the smaller viscosity. The number of the iterations is bounded by 20 for most of the cases and it is independent of the mesh size, viscosity and time step. We chose different time step in this case to test the behavior of the preconditioner. As we notice in the Table 1, 2, 3, the iteration numbers tend to large when the time step is large (small α). The iteration number get stable when time step gets smaller.

Table 1. *Iteration counts of preconditioned GMRES for the unsteady-state Oseen problem with viscosity v = 0.001.*

Grid size	α = 1	α = 10	α = 20	α = 50	α = 100
8 by 8	10	10	11	13	15
16 by 16	12	10	12	14	15
32 by 32	16	11	14	16	16
64 by 64	22	16	14	13	16
128 by 128	27	13	16	16	16

Table 2. *Iteration counts of preconditioned GMRES for the unsteady-state Oseen problem with viscosity v =* 10^{-4}.

Grid size	α = 1	α = 10	α = 20	α = 50	α = 100
8 by 8	11	12	11	12	13
16 by 16	12	13	11	12	14
32 by 32	18	12	13	15	14
64 by 64	27	15	13	14	15
128 by 128	38	13	15	12	14

Table 3. *Iteration counts of preconditioned GMRES for the unsteady-state Oseen problem with viscosity v =* 10^{-5}.

Grid size	α = 1	α = 10	α = 20	α = 50	α = 100
8 by 8	17	16	18	17	13
16 by 16	14	17	16	19	13
32 by 32	17	17	17	15	13
64 by 64	23	16	14	13	14
128 by 128	28	18	15	12	13

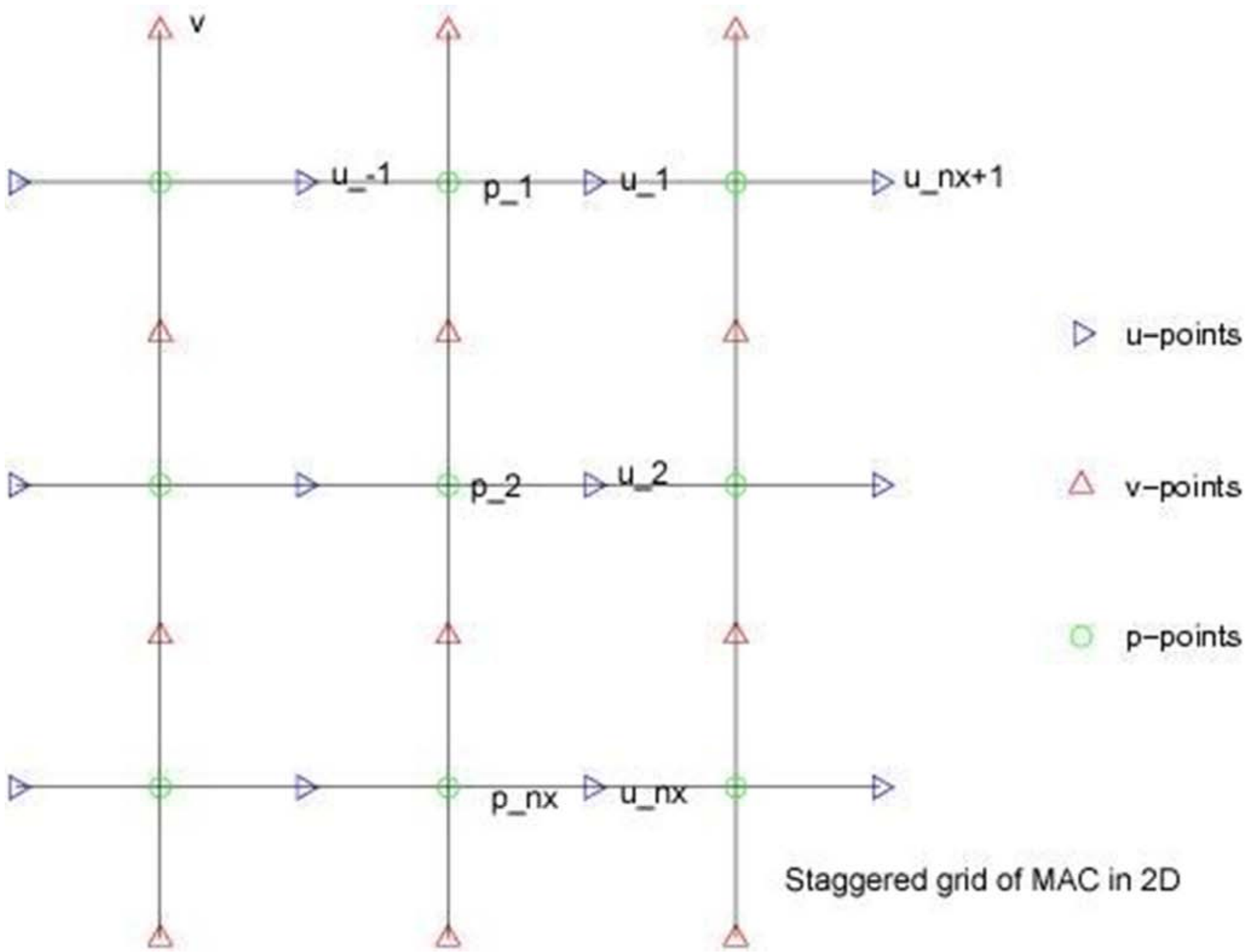

Figure 1. *Staggered Grid of MAC in 2D.*

3.2. The Navier-Stokes Equations Using Finite Elements Discretizaitons

In this experiment, the authors used the software IFISS (Incompressible Flow & Iterative Solver Software) by Silvester, Elamn and Ramage. (see [5], [9]).

The second example we tested Represents Slow Flows in a Rectangular Duct with a Sudden Expansion, or Flow over a Step. The domain is the L-shaped region generated by taking the complement in $(-1, 20) \times (-1,1)$ of the quadrant $(-1,0] \times (-1,0]$. A Poiseuille flow profile is imposed on the inflow boundary $(x = -1;\ 0 \leq y \leq 1)$. For high Reynolds number flow, longer steps are required in order to allow the flow to fully develop. A Neuman condition is applied at the outflow boundary which automatically sets the mean outflow pressure to zero. Figure 2 shows the Stokes flow solution plot. We used the solution of the Stokes flow as the initial approximation of the Picard's iteration. Figure 3 shows the solution plot of the Navier-Stokes equations after the nonlinear iterations, and Figure 4 shows the error plot of the Navier-Stokes flow solution.

We applied the modified HSS preconditioned GMRES methods to solve the linear system at each step of the nonlinear step.

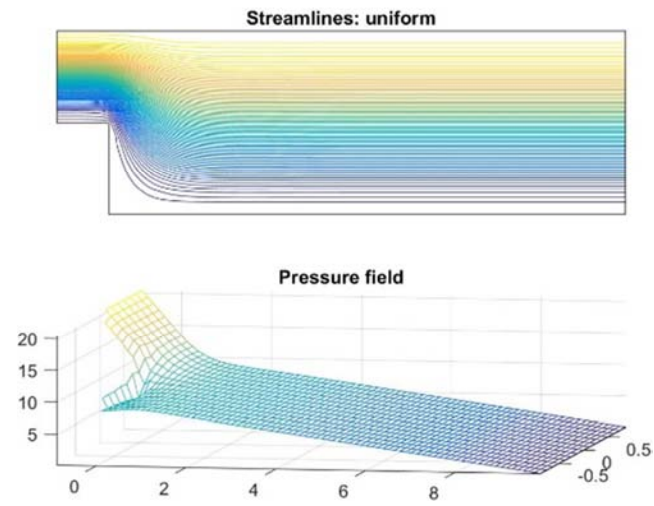

Figure 2. *Stoke flow solution.*

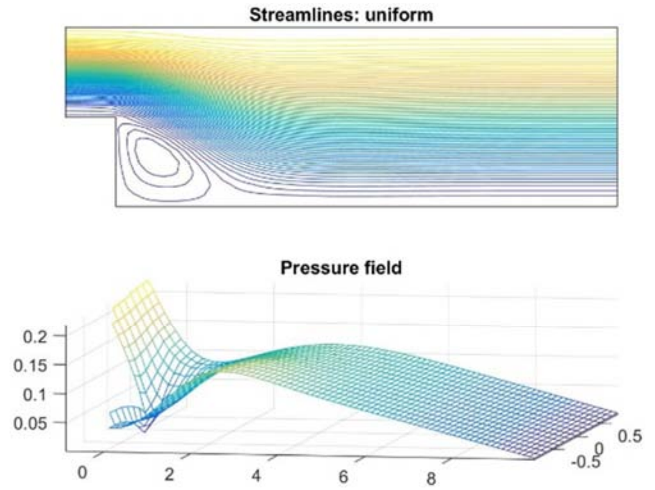

Figure 3. *Navier-Stokes flow solution plot with stabilized Q 2-P 1 approximation.*

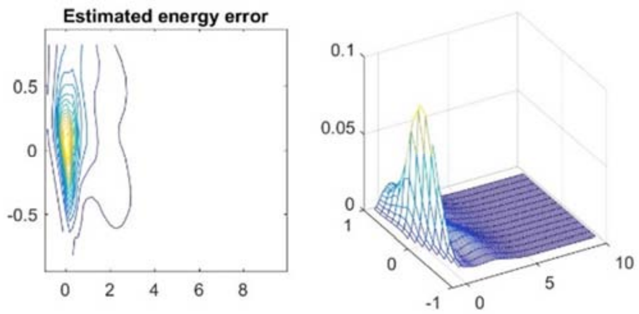

Figure 4. *Estimated error in the computed solution.*

Table 4 is the experimental results for the iteration counts of preconditioned GMRES for the resulting Oseen problem with viscosity 0.001 to 10^{-5}. Table 4 shows the iteration number of GMRES with the viscosity 0.001, 10^{-4} and 10^{-5}. The corresponding Reynolds number is of O (10^3). In each case, we used IFISS to generate the linear system Ax = b. Where A is the discretization of the Naviver-Stokes problem with different viscosity and grid size. The discretization we chose is the Q 2-P 1 approximation. In all table, we can see the iteration number are bounded and it is robust with respect to the grid sizes and viscosity. The iteration numbers increase compared with the unsteady Oseen cases.

Table 4. *Iteration counts of preconditioned GMRES for Navier-Stokes problem with viscosity v = 0.001,*10^{-4}*,* 10^{-5}*.*

Grid size	v = 10^{-3}	v = 10^{-4}	v = 10^{-5}
8 by 8	28	22	21
16 by 16	30	27	22
32 by 32	35	26	22
64 by 64	43	25	23
128 by 128	56	25	22

In the third example, we used the classical test problem in fluid dynamics, known as driven-cavity flow. The models describes a lid moving from left to right in a square domain. A Dirichlet no-flow condition is applied on the side and bottom boundaries. We chose a leaky cavity in this model. That is a nonzero horizontal velocity on the lid is given: $\{y = 1;\ -1\ \leq x\ \leq 1 | u_x = 1\}$.

Figure 5. shows the grid plot of the finite element discretization using Q 1-P 0. Figure 6 is the plot of the computed solutions.

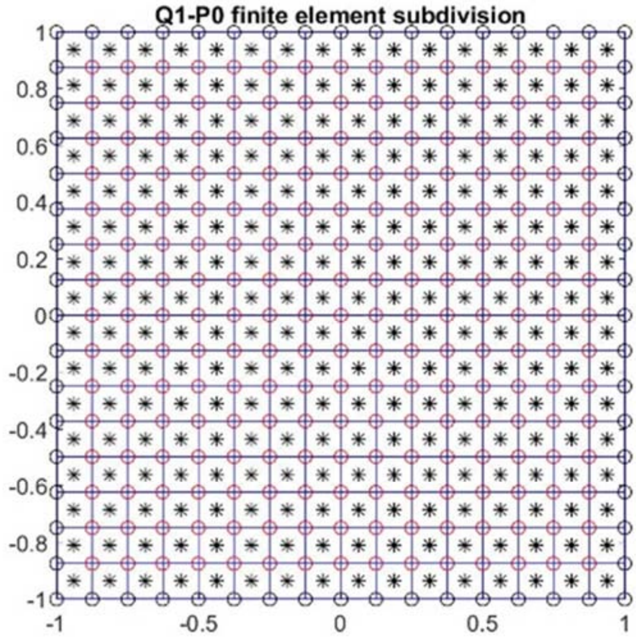

Figure 5. *Q 1-P 0 finite element grid.*

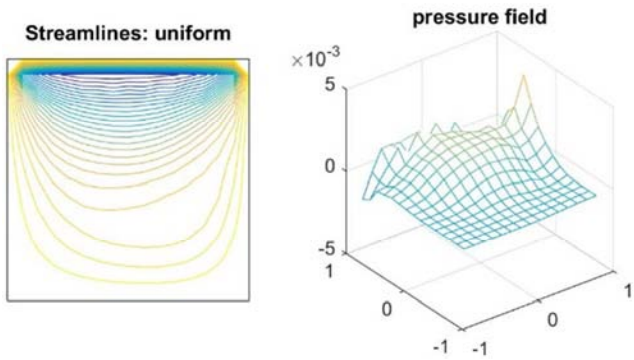

Figure 6. *Computed solution plot of leaky driven-cavity problem.*

Table 4 is the experimental results for the iteration counts of preconditioned GMRES for the resulting Oseen problem with viscosity 0.001 to 10^{-5}. It has the similar behavior with the previous case. We can see that the modified HSS preconditioner works even better for the smaller viscosity.

Table 5. *Iteration counts of preconditioned GMRES for Navier-Stokes problem with viscosity v =* 10^{-3}*.*

Grid size	v = 10^{-3}	v = 10^{-4}	v = 10^{-5}
8 by 8	29	23	22
16 by 16	34	29	23
32 by 32	37	25	22
64 by 64	38	26	23
128 by 128	41	27	23

4. Conclusion

The numerical solver that is reported here has been shown to be a reliable solver of the Navier-Stokes equations with high Reynolds number. We choose the examples with small viscosities varies from 10^{-5} to 10^{-3}. Numerical data shows the robustness of the preconditioned GMRES methods applying to all small viscosity cases. The cost of solving the Hermitian part is low. It can be linear if multigrid method is applied. However, solving the skew-Hermitian part is expensive. With the scaling and shifting, the skew-Hermitian matrix can be diagonal dominant. The choice of the shifting parameter is quite challenging. We are still unclear how to find a mathematical formula to find the optimal ρ. We used numerical test to find the optimal parameters. However it seems like that this optimal parameters has the similar range.

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References

[1] D. Acheson Elementary Fluid Dynamics. Oxford University Press, Oxford, 1990.

[2] G. Batchelor, An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge, 2000.

[3] M. Benzi and J. Liu, An Efficient Solver for the Incompressible Navier-Stokes Equations in Rotation Form, SIAM J. Scientific Computing, 29 (2007), pp. 1959-1981.

[4] H. Elman, D. Silvester, and A. Wathen, Finite Elements and Fast Iterative Solvers with applications in incompressible fluid dynamics. Oxford University Press, Oxford, 2005.

[5] H. Elman, A. Ramage, and D. Silvester, Algorithm 866: IFISS, a Matlab toolbox for modelling incompressible flow, ACM Trans. Math. Softw., 33, 2-14, 2007.

[6] C. A. J. Fletcher, Computational Techniques for Fluid Dynamics, Springer-Verlag, 1988.

[7] M. Lee and R. D. Moser, Direct numerical simulation of turbulent channel flow up to Re $\approx$ 5200, J. Fluid Mech. (2015), vol 774, pp. 395-415. Doi 10.1017/jfm.2015.268.

[8] A. J. Smits and I. Marusic, Wall-bounded turbulence. Phys. Today 66 (9), 25-30, 2013.

[9] D. Silvester, H. Elman and A. Ramage, Incompressible Flow and Iterative Solver Software (IFISS) version 3.5, September, 2016, http://www.manchester.ac.uk/ifiss/

[10] Valerla Simoncini, and Daniel B. Szyld, Theory of inexact Krylov subspace methods and applications to scientific computing, SIAM J. Sci. Comput, 25 (2003), No. 2, pp 454-477.