Applied and Computational Mathematics
2017; 6(4): 196-201
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.20170604.17
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

A Nontrivial Product in the Stable Homotopy of Spheres

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To cite this article:
Wang Chong. A Nontrivial Product in the Stable Homotopy of Spheres. *Applied and Computational Mathematics*. Vol. 6, No. 4, 2017, pp. 196-201. doi: 10.11648/j.acm.20170604.17

Received: June 29, 2017; **Accepted**: July 15, 2017; **Published**: August 7, 2017

Abstract: Let p be an arbitrary odd prime number greater than eleven and A be the mod p Steenrod algebra. In this paper, it has proved that the product $h_0 k_0 \tilde{\delta}_{s+4} \in Ext_A^{s+7,*}(Z_p, Z_p)$ is nontrivial and converges to $\alpha_1 \beta_2 \delta_{s+4}$ nontrivially of order p in $\pi_{q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]-7}S$, where $0 \le s < p-4$, $q = 2(p-1)$ by making use of the Adams spectral sequence.

Keywords: Steenrod Algebra, Cohomology, May Spectral Sequence, Stable Homotopy of Spheres

1. Introduction

Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at an odd prime number p. To determine the stable homotogy groups of spheres $\pi_* S$ is one of the central problems in homotogy theory. One of the main tools to reach it is the Adams spectral sequence (ASS):

$$E_2^{s,t} = Ext_A^{s,t}(Z_p, Z_p) \Rightarrow \pi_{t-s} S,$$

where the $E_2^{s,t}-$ term is the cohomology of A. So far, not so many family $\zeta_{n-1} \neq 0 \in \pi_{p^n q+q-3} S$ for $n \ge 2$ and is represented by $h_0 b_{n-1} \in Ext_A^{3,p^n q+q}(Z_p, Z_p)$ has been detected in [1] in the ASS, where $q = 2(p-1)$.

To determine the stable homotopy groups of spheres is one of the most important problems in algebraic topology. So far, several methods have been found to determine the stable homotopy groups of spheres. For example, we have the classical Adams spectral sequence (ASS) (cf. [2]) based on the Eilenberg-MacLane spectrum KZ_p, whose E_2-term is $Ext_A^{s,t}(Z_p, Z_p)$ and the Adams differential is given by

$$\tilde{d}_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

where A denotes the mod p Steenrod algebra.

There are three problems in using the ASS: calculation of E_2-term $Ext_A^{*,*}(Z_p, Z_p)$, computation of the differentials and determination of the nontrivial extensions from E_∞ to the stable homotopy groups of spheres. So, for computing the stable homotopy groups of spheres with the classical ASS, we must compute the E_2-term of the ASS, $Ext_A^{*,*}(Z_p, Z_p)$.

Throughout this paper, p denotes an odd prime and $q = 2(p-1)$. The known results on $Ext_A^{*,*}(Z_p, Z_p)$ are as follows. $Ext_A^{0,*}(Z_p, Z_p)$ is trivial by its definition. From [3], $Ext_A^{1,*}(Z_p, Z_p)$ has Z_p-basis consisting of

$$a_0 \in Ext_A^{1,1}(Z_p, Z_p),\ h_i \in Ext_A^{1,p^i q}(Z_p, Z_p)$$

for all $i \ge 0$ and $Ext_A^{2,*}(Z_p, Z_p)$ has Z_p- basis consisting of α_2, a_0^2, $a_0 h_i (i > 0)$, $g_i (i \ge 0)$, $k_i (i \ge 0)$, $b_i (i \ge 0)$ and $h_i h_j (j \ge i+2, i \ge 0)$ whose internal degrees are $2q+1$, 2, $p^i q+1$, $p^{i+1}q+2p^i q$, $2p^{i+1}q+p^i q$ and $p^i q+p^j q$ respectively.

Let M be the Moore spectrum modulo a prime number $p \ge 3$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \sum S.$$

Let $\alpha:\sum^q M\to M$ be the Adams map and K be its cofibre given by the cofibration

$$\sum^q M\xrightarrow{\alpha}M\xrightarrow{i'}K\xrightarrow{j'}\sum^{q+1}M,$$

where $q=2(p-1)$. This spectrum which we briefly write as K is known to be the Toda-Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta:\sum^{(p+1)q}K\to K$ given by the cofibration

$$\sum^{(p+1)q}K\xrightarrow{\beta}K\xrightarrow{\bar{i}}V(2)\xrightarrow{\bar{j}}\sum^{(p+1)q+1}K.$$

Let $\gamma:\sum^{q(p^2+p+1)}V(2)\to V(2)$ be the v_3-map. As we know, for $t>0$, the $\alpha-$ element $\alpha_t=j\alpha^t i$, the $\beta-$ element $\beta_t=jj'\beta^t i'i$, the $\gamma-$ element $\gamma_t=jj'\bar{j}\gamma^t\bar{i}i'i$.

In [4], we have the following results:

(1) For $p\ge 3$ and $t\ge 1,\alpha_t\ne 0$ in π_*S.

(2) For $p\ge 5$ and $t\ge 1,\beta_t\ne 0$ in π_*S.

(3) For $p\ge 7$ and $t\ge 1,\gamma_t\ne 0$ in π_*S.

Studying higher-dimensional cohomology of the mod p Steenrod algebra A is an interesting subject and studied by several authors. For example, In 1980, Aikawa [5] determined $Ext_A^{3,*}(Z_p,Z_p)$ by $\lambda-$ algebra. Liu and Zhao [6] prove the following theorem.

Theorem 1.1 For $p\ge 11$ and $4\le s<p$, the product $h_0b_0\tilde{\delta}_s\ne 0$ in the classical Adams spectral sequence, where $\tilde{\delta}_s$ is given in [7].

2. The May Spectral Sequence

The most successful method to compute $Ext_A^{*,*}(Z_p,Z_p)$ is the MSS. From [8], there is a May spectral sequence (MSS) $\{E_r^{s,t,*},d_r\}$ which converges to $Ext_A^{s,t}(Z_p,Z_p)$ with E_1- term

$$E_1^{*,*,*}=E(h_{m,i}\,|m>0,i\ge 0)\otimes P(b_{m,i}\,|m>0,i\ge 0)\otimes P(a_n\,|n\ge 0)\quad (1)$$

where $E(\ )$ is the exterior algebra, $P(\ )$ is the polynomial algebra, and $h_{m,i}\in E_1^{1,2(p^m-1)p^i,2m-1},b_{m,i}\in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)},a_n\in E_1^{1,2p^n-1,2n+1}$

One has

$$d_r:E_r^{s,t,u}\to E_r^{s+1,t,u-r}\quad (2)$$

and if $x\in E_r^{s,t,*}$ and $y\in E_r^{s',t',*}$, then

$$d_r(x.y)=d_r(x)\cdot y+(-1)^s x\cdot d_r(y)\quad (3)$$

In particular, the first May differential d_1 is given by

$$d_1(h_{i,j})=\sum_{0<k<i}h_{i-k,k+j}h_{k,j},$$

$$d_1(a_i)=\sum_{0\le k<i}h_{i-k,k}a_k,\quad d_1(b_{i,j})=0\quad (4)$$

There also exists a graded commutativity in the MSS:

$$x\cdot y=(-1)^{ss'+tt'}y\cdot x\quad \text{for}\quad x,y=h_{m,i},b_{m,i}\text{ or }a_n.$$

For each element $x\in E_1^{s,t,u}$, we define $\dim x=s$, $\deg x=t,M(x)=u$. Then we have that

$$\begin{cases}\dim h_{i,j}=\dim a_i=1,\\ \dim b_{i,j}=2,\deg a_0=1,\\ \deg h_{i,j}=q(p^{i+j-1}+\ldots+p^j),\\ \deg b_{i,j}=q(p^{i+j}+\ldots+p^{j+1}),\\ \deg a_i=q(p^{i-1}+\ldots+1)+1,\\ M(h_{i,j})=M(a_{i-1})=2i-1,\\ M(b_{i,j})=(2i-1)p,\end{cases}\quad (5)$$

where $i\ge 1,j\ge 0$.

Note that by the knowledge on the $p-$ adic expression in number theory, for each integer $t\ge 0$, it can be expressed uniquely as $t=q(c_np^n+c_{n-1}p^{n-1}+\ldots+c_1p+c_0)+e$ where $0\le c_i<p(0\le i<n),p>c_n>0,0\le e<q.$

For the convenience of writing, we make the following rules:

(i) If $i>j$, we put a_i on the left side of a_j;

(ii) If $j<k$, we put $h_{i,j}$ on the left side of $h_{w,k}$;

(iii) If $i>w$, we put $h_{i,j}$ on the left side of $h_{w,j}$;

(iv) Apply the rules (ii) and (iii) to $b_{i,j}$.

3. Proof of the Main Theorem

Before showing the main theorem, we first give some important lemmas which will be used in the proof of it. The first one is a lemma on the representative of $\tilde{\delta}_{s+4}$ in the May spectral sequence.

Lemma 3.1 [7] For $p\ge 11$ and $0\le s<p-4$, then the fourth Geerk letter element $\tilde{\delta}_{s+4}\in Ext_A^{s+4,t_1(s)}(Z_p,Z_p)$ is represented by

$$a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_1^{s+4,t_1(s),*}$$

in the E_1- term of the May spectral sequence, where $\tilde{\delta}_{s+4}$ is actually $\tilde{\alpha}_{s+4}^{(4)}$ and

$$t_1(s)=q[(s+1)+(s+2)p+(s+3)p^2+(s+4)p^3)+s.$$

By (2), we know that to prove the non-triviality of the product $h_0k_0\tilde{\delta}_{s+4}$, we have to show that the representative of the product cannot be hit any May differential. For doing it, we give the following two lemma.

Lemma 3.2 Let $p \geq 11$ and $0 \leq s < p-4$. Then we have the May E_1 − term $E_1^{s+6,t(s),*} = 0$, where

$$t(s) = q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3)+s.$$

Now we give the proof of the above lemma.

Proof Consider $h = x_1x_2...x_m \in E_1^{s+6,t(s),*}$ in the MSS, where x_i is one of $a_k, h_{r,j}$ or $b_{u,z}$, $0 \leq k \leq 4$, $0 \leq r+j \leq 4$, $0 \leq u+z \leq 3$, $r>0, j \geq 0$, $u>0$, $z \geq 0$. By (5) we can assume that

$$\deg x_i = q(c_{i,3}p^3 + c_{i,2}p^2 + c_{i,1}p + c_{i,0}) + e_i$$

where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_k$, or $e_i = 0$. It follows

that dim $h = \sum_{i=1}^{m} \dim x_i = s+6$ and

$$\deg h = \sum_{i=1}^{m} \deg x_i$$

$$= q[(\sum_{i=1}^{m} c_{i,3})p^3 + (\sum_{i=1}^{m} c_{i,2})p^2 + (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})] + (\sum_{i=1}^{m} e_i)$$

$$= q[(s+4)p^3 + (s+3)p^2 + (s+4)p + (s+3)] + s$$

Note that $\dim h_{i,j} = \dim a_i = 1$, $\dim b_{i,j} = 2$, and $0 \leq s < p-4$. From $\dim h = \sum_{i=1}^{m} \dim x_i = s+6$, we can have

$$m \leq s+6 < p+2.$$

Using $0 \leq s+4, s+3, s < p$ and the knowledge on the p-adic expression in number theory, we have that

$$\begin{cases} \sum_{i=1}^{m} e_i = s; \\ \sum_{i=1}^{m} c_{i,0} = s+3; \\ \sum_{i=1}^{m} c_{i,1} = s+4; \\ \sum_{i=1}^{m} c_{i,2} = s+3; \\ \sum_{i=1}^{m} c_{i,3} = s+4; \end{cases} \tag{6}$$

By $c_{i,2} = 0$ or 1, one has $m \geq s+4$ from $\sum_{i=1}^{m} c_{i,3} = s+4$.

Note that $m \leq s+6$. Thus m may equal $s+4, s+5, s+6$.

Since $\sum_{i=1}^{m} e_i = s$, $\deg h_{i,j} \equiv 0 (\mathrm{mod}\, q)$ $(i>0, j \geq 0)$, $\deg a_i \equiv 1 (\mathrm{mod} q)(i \geq 0)$ and $\deg b_{i,j} \equiv 0 (\mathrm{mod}\, q)$ $(i>0, j \geq 0)$, then by the graded commutativity of $E_1^{*,*,*}$ and degree reasons, we can assume that $h = a_0^x a_1^y a_2^z a_3^k a_4^l h'$ with $h' = x_{s+1}x_{s+2}...x_m$, where

$$0 \leq x, y, z, k, l \leq s, \ x+y+z+k+l = s.$$

Consequently, we have $h' = x_{s+1}x_{s+2}...x_m \in E_1^{6,t_2(s),*}$, where

$$t_2(s) = q[(s+4-l)p^3 + (s+3-l-k)p^2 +$$
$$(s+4-l-k-z)p + (s+3-l-k-z-y)].$$

Form (6) we have

$$\begin{cases} \sum_{i=s+1}^{m} e_i = 0; \\ \sum_{i=s+1}^{m} c_{i,0} = s+3-l-k-z-y; \\ \sum_{i=s+1}^{m} c_{i,1} = s+4-l-k-z; \\ \sum_{i=s+1}^{m} c_{i,2} = s+3-l-k; \\ \sum_{i=s+1}^{m} c_{i,3} = s+4-l; \end{cases} \tag{7}$$

By the reason of dimension, all the possibilities of h' can be listed as $y_1...y_6$, $y_1...y_4z_1$, $y_1y_2z_1z_2$, $z_1z_2z_3$ where the y_i is in the form of $h_{r,j}$, $0 \leq r+j \leq 4$, $r>0, j \geq 0$ and z_i is in the form of $b_{u,z}$, $0 \leq u+z \leq 3$, $u>0$, $z \geq 0$.

Case 1. $m = s+4$. From $\sum_{i=s+1}^{s+4} c_{i,3} = s+4-l$ in (7), we have that $l = s+4-\sum_{i=s+1}^{s+4} c_{i,3} \geq s$. Thus $l = s$, x=y=z=k=0.

So $h' = x_{s+1}x_{s+2}x_{s+3}x_{s+4} \in E_1^{6,q(4p^3+3p^2+4p+3),*}$, and it must be in the form of $y_1y_2z_1z_2$. But because the constant coefficient in $t_2(s)$ is 3, h' is impossible to exist in this case. Then h doesn't exist either.

Case 2. $m = s+5$. From $\sum_{i=s+1}^{s+5} c_{i,3} = s+4-l$ in (7), we have that $l = s+4-\sum_{i=s+1}^{s+5} c_{i,3} \geq s-1$. Thus $l = s-1$ or s, and

$$h' = y_1 ... y_4 z_1 \in E_1^{6,t_2(s),*}.$$

(1) If $l = s-1$, then

$$t_2(s) = q[5p^3 + (\ )p^2 + (\ )p + (s+3-l-k-z-y)]$$

Because the coefficient of p^3 is 5, $h' = y_1 ... y_4 z_1 \in E_1^{6,t_2(s),*}$, $h_{4,0}, h_{3,1}, h_{2,2}, h_{1,3}$ and some $b_{u,z}$ ($u+z=3$, $u>0$, $z \geq 0$.) must be contained in h'. So the constant coefficient in $t_2(s)$ should be 1, this is conflict with $s+3-l-k-z-y \geq 3$. h' is impossible to exist and then h doesn't exist either.

(2) If $l = s$, then $t_2(s) = q[4p^3 + 3p^2 + 4p + 3]$.

Because $h' = y_1 ... y_4 z_1 \in E_1^{6,t_2(s),*}$, the coefficient of p^3 is 4 and the constant coefficient in $t_2(s)$ is 3, no matter whether the equation of $u+z=3$ in $z_1 = b_{u,z}$ is established or not, h' is impossible to exist in this case and then h doesn't exist either.

Finally, in order to express this result more intuitively, we list all the possibilities of case 2 in the following table 1.

Table 1. *m=s+5.*

possibility	l	x	y	z	k	$E_1^{6,t_2(s),*}$	h'
The 1st	s-1	1	0	0	0	$E_1^{6,q(5p^3+4p^2+5p+4),*} = 0$	Nonexistence
The 2nd	s-1	0	1	0	0	$E_1^{6,q(5p^3+4p^2+5p+3),*} = 0$	Nonexistence
The 3rd	s-1	0	0	1	0	$E_1^{6,q(5p^3+4p^2+4p+3),*} = 0$	Nonexistence
The 4th	s-1	0	0	0	1	$E_1^{6,q(5p^3+3p^2+4p+3),*} = 0$	Nonexistence
The 5th	s	0	0	0	0	$E_1^{6,q(4p^3+3p^2+4p+3),*} = 0$	Nonexistence

Case 3. $m = s+6$. From $\sum_{i=s+1}^{s+6} c_{i,3} = s+4-l$ in (7), we have that $l = s+4-\sum_{i=s+1}^{s+6} c_{i,3} \geq s-2$. Thus $l = s-2$, $s-1$ or s, and $h' = y_1 ... y_6 \in E_1^{6,t_2(s),*}$. The coefficient of p^3 in $t_2(s)$ is greater than or equal to 4. So $h_{4,0}, h_{3,1}, h_{2,2}, h_{1,3}$ must be contained in h'.

When $l = s-2$ or $s-1$, the coefficient of p^3 in $t_2(s)$ is 6 or 5 and $h' = y_1 ... y_6 \in E_1^{6,t_2(s),*}$ is impossible to exist. h doesn't exist either.

When $l = s$, $t_2(s) = q[4p^3 + 3p^2 + 4p + 3]$. Because $h' = y_1 ... y_6 \in E_1^{6,t_2(s),*}$, $h_{4,0}, h_{3,1}, h_{2,2}, h_{1,3}$ must be contained in h' similarly.

But $\deg\{h_{4,0}\, h_{3,1} h_{2,2} h_{1,3}\} = q[4p^3 + 3p^2 + 2p + 1]$, in this case h' is impossible to exist. Then h doesn't exist either.

We list all the possibilities in the following table.

Table 2. *m=s+6.*

possibility	l	x	y	z	k		h'
The 1st	s-2	2	0	0	0	$E_1^{6,q(6p^3+5p^2+6p+5),*} = 0$	Nonexistence
The 2nd	s-2	0	2	0	0	$E_1^{6,q(6p^3+5p^2+6p+3),*} = 0$	Nonexistence
The 3rd	s-2	0	0	2	0	$E_1^{6,q(6p^3+5p^2+4p+3),*} = 0$	Nonexistence
The 4th	s-2	0	0	0	2	$E_1^{6,q(6p^3+3p^2+4p+3),*} = 0$	Nonexistence
The 5th	s-2	1	1	0	0	$E_1^{6,q(6p^3+5p^2+6p+4),*} = 0$	Nonexistence
The 6th	s-2	1	0	1	0	$E_1^{6,q(6p^3+5p^2+5p+4),*} = 0$	Nonexistence
The 7th	s-2	1	0	0	1	$E_1^{6,q(6p^3+4p^2+5p+4),*} = 0$	Nonexistence

possibility	l	x	y	z	k		h'
The 8th	s-2	0	1	1	0	$E_1^{6,q(6p^3+5p^2+5p+3),*}=0$	Nonexistence
The 9th	s-2	0	1	0	1	$E_1^{6,q(6p^3+4p^2+5p+3),*}=0$	Nonexistence
The 10th	s-2	0	0	1	1	$E_1^{6,q(6p^3+4p^2+4p+3),*}=0$	Nonexistence
The 11th	s-1	1	0	0	0	$E_1^{6,q(5p^3+4p^2+5p+4),*}=0$	Nonexistence
The 12th	s-1	0	1	0	0	$E_1^{6,q(5p^3+4p^2+5p+3),*}=0$	Nonexistence
The 13th	s-1	0	0	1	0		Nonexistence
The 14th	s-1	0	0	0	1	$E_1^{6,q(5p^3+3p^2+4p+3),*}=0$	Nonexistence
The 15th	s	0	0	0	0		Nonexistence

According to above mentioned analysis, Lamma 3.2 follows.

Lemma 3.3 Let $p\geq 11$ and $0\leq s<p-4$ then in the cohomology of the mod p Steenrod algebra A, the product

$$h_0k_0\tilde{\delta}_{s+4}\in Ext_A^{s+7,t(s)}(Z_p,Z_p)$$

is nontrivial, where

$$t(s)=q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3]+s\,.$$

Proof Since $h_{1,0}$, $h_{2,0}h_{1,1}$ and $a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to h_0, k_0, $\tilde{\delta}_{s+4}\in Ext_A^{*,*}(Z_p,Z_p)$ for $n\geq 0$, so

$$h_{1,0}h_{2,0}h_{1,1}a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_1^{s+7,t(s),*}$$

is a permanent cycle in the MSS and converges to

$$h_0k_0\tilde{\delta}_{s+4}\in Ext_A^{s+7,*}(Z_p,Z_p)\,.$$

From Lamma 3.2, we see that $E_1^{s+6,t(s),*}=0$, where

$$t(s)=q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3)+s\,.$$

Then we can have

$$E_r^{s+6,q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3],*}=0\quad\text{for}\quad r\geq 1\,.$$

Thus the permanent cycle

$$h_{1,0}h_{2,0}h_{1,1}a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_r^{s+7,t(s),*}$$

does not bound. That is to say,

$$h_{1,0}h_{2,0}h_{1,1}a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_r^{s+7,t(s),*}$$

can not be hit by any differential in the MSS. It follows that

$$h_{1,0}h_{2,0}h_{1,1}a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_r^{s+7,t(s),*}$$

is a permanent cycle in the May spectral sequence and converge nontrivially to $h_0k_0\tilde{\delta}_{s+4}\in Ext_A^{s+7,*}(Z_p,Z_p)$.

It follows that

$$h_0k_0\tilde{\delta}_{s+4}\neq 0\in Ext_A^{s+7,q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3]+s,*}(Z_p,Z_p)$$

and the theorem is proved.

Lemma 3.4 Let $p\geq 11$, $0\leq s<p-4$ and $2\leq r\leq s+7$, then the groups

$$Ext_A^{s+7-r,q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3]+s-r+1}(Z_p,Z_p)=0\,.$$

Proof The proof is divided into two parts.

Case 1. $r=s+7$. In this case,

$$Ext_A^{s+7-r,q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3]+s-r+1}(Z_p,Z_p)=0$$

by $q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3]-6>0$ (cf. [3]).

Case 2. $2\leq r\leq s+6$.

To prove that $Ext_A^{s+7-r,t''}(Z_p,Z_p)=0$, it suffices to prove that $E_1^{s+7-r,t'',*}=0$ in the MSS [9, 10], where

$$t''=q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3]+s-r+1$$

Consider $h=x_1x_2\ldots x_m\in E_1^{s+7-r,t'',*}$ in the MSS, where x_i is one of $a_k, h_{r,j}$ or $b_{u,z}$, $0\leq k\leq 4$, $0\leq r+j\leq 4$, $0\leq u+z\leq 3$, $r>0, j\geq 0$, $u>0$, $z\geq 0$. Assume that

$$\deg\ x_i=q(c_{i,3}p^3+c_{i,2}p^2+c_{i,1}p+c_{i,0})+e_i$$

where $c_{i,j}=0$ or 1, $e_i=1$ if $x_i=a_k$, or $e_i=0$. It follows that dim $h=\sum_{i=1}^m\dim x_i=s+7-r$ and

$$\begin{aligned}\deg h&=\sum_{i=1}^m\deg x_i\\&=q[(\sum_{i=1}^m c_{i,3})p^3+(\sum_{i=1}^m c_{i,2})p^2+(\sum_{i=1}^m c_{i,1})p+(\sum_{i=1}^m c_{i,0})]+(\sum_{i=1}^m e_i)\\&=q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+s-r+1\end{aligned}$$

Note that $\dim x_i = 1$ or 2, we get that

$$m \le \dim h = s+7-r \le s+5 \le p-5+5 = p.$$

By the knowledge about $p-$ adic expression in number theory, we have

$$\sum_{i=1}^{m} c_{i,3} = s+4, \sum_{i=1}^{m} c_{i,2} = s+3, \sum_{i=1}^{m} c_{i,1} = s+4,$$

$$\sum_{i=1}^{m} c_{i,0} = s+3, \sum_{i=1}^{m} e_i = s-r+1.$$

When $r > 3$, we can get $\dim h = s+7-r < s+4$.

Meanwhile, we have $m \ge s+4$ from $\sum_{i=1}^{m} c_{i,3} = s+4$, so $\dim h \ge s+4$. This is a contradiction.

When $r = 3$, we can get $\dim h = s+7-r = s+4$. By an argument similar to that used in the proof of Lamma 3.1,
we can show that

$$E_1^{s+4,q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+(s-2),*} = 0.$$

When $r = 2$, then we can get $\dim h = s+5$, $m \le s+5$. But from $\sum_{i=1}^{m} c_{i,3} = s+4$, we get that $m \ge s+4$. There are two possiblities: $m = s+4$ or $m = s+5$. By an argument similar to that used in the proof of lamma 3.2, we can prove that $E_1^{s+4,q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]+(s-1),*} = 0$.

4. Conclusion

Theorem Let $p \ge 11$, then in the cohomology of the mod p Steenrod algebra A, the product

$$h_0 k_0 \tilde{\delta}_{s+4} \ne 0 \in Ext_A^{s+7,t(s)}(Z_p, Z_p)$$

is a permanent cycle in the ASS and converges to $\alpha_1 \beta_2 \delta_{s+4}$ nontrivially of order p in

$$\pi_{q[(s+4)p^3+(s+3)p^2+(s+4)p+(s+3)]-7} S,$$

where $0 \le s < p-4$, $q = 2(p-1)$.

Proof First, by Lamma 3.3 we know that

$$h_0 k_0 \tilde{\delta}_{s+4} \ne 0 \in Ext_A^{s+7,q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3]+s,*}(Z_p, Z_p)$$

Next, it is known that $h_0, k_0 \in Ext_A^{*,*}(Z_p, Z_p)$ converge nontrivially to the $\alpha-$ element α_1 and the $\beta-$ element β_2 in the ASS respctivly. The $\delta-$ element δ_{s+4} is represented by $\tilde{\delta}_{s+4} \in Ext_A^{s+4,q[(s+4)p^3+(s+3)p^2+(s+2)p+(s+1)]+s}(Z_p, Z_p)$ in the ASS (cf [11]). Then the map $\alpha_1 \beta_2 \delta_{s+4} \in \pi_* S$ is represented by $h_0 k_0 \tilde{\delta}_{s+4} \ne 0 \in Ext_A^{s+7,t(s)}(Z_p, Z_p)$, where

$$t(s) = q[(s+3)+(s+4)p+(s+3)p^2+(s+4)p^3)+s.$$

Acknowledgements

The reseach was supported by the Youth Foundation of Hebei Educational Committee (QN 2017505). We thank the referees for their time and comments.

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