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Volterra Integral Equations with Vanishing Delay

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Abstract: In this article, we use a Chebyshev spectral-collocation method to solve the Volterra integral equations with vanishing delay. Then a rigorous error analysis provided by the proposed method shows that the numerical error decay exponentially in the infinity norm and in the Chebyshev weighted Hilbert space norm. Numerical results are presented, which confirm the theoretical predicition of the exponential rate of convergence.

Keywords: Chebyshev Spectral-Collocation Method, Volterra Integral Equations, Vanishing Delay, Error Estimate, Convergence Analysis

1. Introduction

The Volterra integral equations (VIES) arise in many modeling problems in mathematical physics and chemical reaction, such as in the heat conduction, potential theory, fluid dynamic and radiative heat transfer problems. There are many methods to solve VIES, such as Legendre spectral-collocation method [1], Jacobi spectral-collocation method [2], spectral Galerkin method [3,4], Chebyshev spectral-collocation method [5] and so on. In this paper, according to [5], we use a Chebyshev spectral-collocation method, where the collocation points are Chebyshev Gauss, Chebyshev Gauss-Radau, Chebyshev Gauss-Lobatto points to solve Volterra integral equations with vanishing delay.

2. Definition

The Volterra integral equations with the vanishing delay are defined as

$$y(\tau)=f(\tau)+\int_0^{\tau} R_1(\tau,\xi)\,y(\xi)\,d\xi+\int_0^{q\tau^2} R_2(\tau,\eta)\,y(\eta)\,d\eta,\quad \tau\in[0,T], \tag{1}$$

where $0<T<+\infty$, and

$$f(\tau)\in C^m([0,T]),\quad R_1(\tau,\xi)\in C^m(\Omega_1),\ R_2(\tau,\eta)\in C^m(\Omega_2),\quad m\geq 1, \tag{2}$$

where $\Omega_1:=\{(\tau,\xi):0\le\xi\le\tau\le T\}$, $\Omega_2:=\{(\tau,\eta):0\le\eta\le q\tau^2\le\tau\le T\}$, and q is a constant, $0<q<1$.

3. Chebyshev Spectral-Collocation Method

For ease of analysis, we will transfer the integral interval $[0,\tau]$ and $[0,q\tau^2]$ to fixed interval [-1, 1].

3.1. The Changes of Variables

Firstly, we use the changes of variables

$$\tau=\frac{T}{2}(1+x),\quad x:=\frac{2}{T}\tau-1,$$

$$\xi=\frac{T}{2}(1+s),\quad s:=\frac{2}{T}\xi-1,$$

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$$\eta = \frac{T}{2}(1+t), \quad t := \frac{2}{T}\eta - 1.$$

Then (1) becomes

$$u(x) = g(x) + \int_{-1}^{x} \tilde{K}_1(x,s)u(s)\,ds + \int_{-1}^{\frac{qT}{2}(1+x)^2-1} \tilde{K}_2(x,t)u(t)\,dt, \quad x \in [-1,1], \quad (3)$$

where $u(x) := y\left(\frac{T}{2}(1+x)\right)$, $g(x) := f\left(\frac{T}{2}(1+x)\right)$,

$$\tilde{K}_1(x,s) := \frac{T}{2}R_1\left(\frac{T}{2}(1+x), \frac{T}{2}(1+s)\right),$$

$$\tilde{K}_2(x,t) := \frac{T}{2}R_2\left(\frac{T}{2}(1+x), \frac{T}{2}(1+t)\right).$$

3.2. Set the Collocation Points

Now we assume that $\{x_i\}_{i=0}^{N}$ (see,e.g., [6]) are the set of $N+1-$point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points, then (3) holds at x_i :

$$u(x_i) = g(x_i) + \int_{-1}^{x_i} \tilde{K}_1(x_i,s)u(s)\,ds + \int_{-1}^{\frac{qT}{2}(1+x_i)^2-1} \tilde{K}_2(x_i,t)u(t)\,dt, \; 0 \le i \le N. \quad (4)$$

3.3. Linear Transformation

We make two simple linear transformations

$$s_x(z) = \frac{x+1}{2}z + \frac{x-1}{2}, \quad z \in [-1,1],$$

$$t_x(v) = \frac{qT}{4}(1+x)^2 v + \frac{qT}{4}(1+x)^2 - 1, \quad v \in [-1,1].$$

Then (4) becomes

$$u(x_i) = g(x_i) + \int_{-1}^{1} K_1(x_i,z)u\left(s_{x_i}(z)\right)dz + \int_{-1}^{1} K_2(x_i,v)u\left(t_{x_i}(v)\right)dv, \quad i = 0,1,\cdots N, \quad (5)$$

where $K_1(x,z) := \frac{x+1}{2}\tilde{K}_1(x, s_x(z))$, $K_2(x,v) := \frac{qT(1+x)^2}{4}\tilde{K}_2(x, t_x(v))$.

Applying appropriate $N+1-$point Gauss quadrature formula, we can obtain that

$$u(x_i) \approx g(x_i) + \sum_{k=0}^{N} K_1(x_i,z_k)u\left(s_{x_i}(z_k)\right)\omega_k + \sum_{k=0}^{N} K_2(x_i,v_k)u\left(t_{x_i}(v_k)\right)\omega_k, \quad i = 0,1,\cdots N,$$

where $z_k = v_k$ are the $N+1-$point Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto points, corresponding weight ω_k, $k = 0,1,\cdots N$ (see, e.g.[6]). We use u_i to approximate the function value $u(x_i)$ and use $u^N(x) := \sum_{j=0}^{N} u_j F_j(x)$ to approximate the function $u(x)$, where $F_j(x)$ is the $j-th$ Lagrange basic function.

First, deal with $\sum_{k=0}^{N} K_1(x_i,z_k)u\left(s_{x_i}(z_k)\right)\omega_k$,

$$\begin{aligned}
&\sum_{k=0}^{N} K_1(x_i,z_k)u\left(s_{x_i}(z_k)\right)\omega_k \\
&= \sum_{k=0}^{N} K_1(x_i,z_k)\sum_{j=0}^{N} u_j F_j\left(s_{x_i}(z_k)\right)\omega_k \\
&= \sum_{k=0}^{N} K_1(x_i,z_k)\left(u_0 F_0\left(s_{x_i}(z_k)\right)\omega_k + \cdots + u_N F_N\left(s_{x_i}(z_k)\right)\omega_k\right) \\
&= \sum_{k=0}^{N} K_1(x_i,z_k)u_0 F_0\left(s_{x_i}(z_k)\right)\omega_k + \cdots + \sum_{k=0}^{N} K_1(x_i,z_k)u_N F_N\left(s_{x_i}(z_k)\right)\omega_k \\
&= K_1(x_i,z_0)u_0 F_0\left(s_{x_i}(z_0)\right)\omega_0 + \cdots + K_1(x_i,z_0)u_N F_N\left(s_{x_i}(z_0)\right)\omega_0 \\
&\quad + K_1(x_i,z_1)u_0 F_0\left(s_{x_i}(z_1)\right)\omega_1 + \cdots + K_1(x_i,z_1)u_N F_N\left(s_{x_i}(z_1)\right)\omega_1 + \cdots \\
&\quad + K_1(x_i,z_N)u_0 F_0\left(s_{x_i}(z_N)\right)\omega_N + \cdots + K_1(x_i,z_N)u_N F_N\left(s_{x_i}(z_N)\right)\omega_N
\end{aligned}$$

$$= u_0 \sum_{k=0}^{N} K_1(x_i, z_k) F_0(s_{x_i}(z_k)) \omega_k + \cdots + u_N \sum_{k=0}^{N} K_1(x_i, z_k) F_N(s_{x_i}(z_k)) \omega_k$$
$$= \sum_{j=0}^{N} u_j \sum_{k=0}^{N} K_1(x_i, z_k) F_j(s_{x_i}(z_k)) \omega_k .$$

Similarly, $\sum_{k=0}^{N} K_2(x_i, \mathrm{v}_k) u(t_{x_i}(v_k)) \omega_k = \sum_{j=0}^{N} u_j \sum_{k=0}^{N} K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \omega_k$,

then

$$u_i = g(x_i) + \sum_{j=0}^{N} u_j \sum_{k=0}^{N} \omega_k \left(K_1(x_i, z_k) F_j(s_{x_i}(z_k)) + K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \right), \quad i = 0, 1, \cdots N. \tag{6}$$

The Chebyshev spectral-collocation method is to seek $u_i = g(x_i) + \sum_{j=0}^{N} u_j \sum_{k=0}^{N} \omega_k \left(K_1(x_i, z_k) F_j(s_{x_i}(z_k)) + K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \right), \ i = 0, 1, \cdots N.$ $u^N(x)$ such that $\{u_i\}_{i=0}^N$ satisfies the above equation.

3.4. Implementation of the Spectral Collocation Algorithm

(6) can be written in matrix form: $U = G + AU$,

where $U := [u_0, u_1, \cdots, u_N]'$,

$$G := [g(x_0), g(x_1), \cdots, g(x_N)]',$$

$$A := (a_{ij})_{(N+1)\times(N+1)},$$

$$a_{ij} := \sum_{k=0}^{N} \omega_k \left(K_1(x_i, z_k) F_j(s_{x_i}(z_k)) + K_2(x_i, v_k) F_j(t_{x_i}(v_k)) \right).$$

We now discuss an efficient computation of $F_j(s)$ .Considering Chebyshev function

$$F_j(s) = \sum_{p=0}^{N} a_{p,j} T_p(s), \tag{7}$$

where $a_{p,j}$ is called the discrete polynomial coefficients of F_j. The inverse relation (see, e.g., [6]) is

$$a_{p,j} = \frac{1}{\gamma_p} \sum_{i=0}^{N} F_j(x_i) T_p(x_i) \omega_i^c = T_p(x_i) \omega_j^c / \gamma_p, \tag{8}$$

and ω_j^c is the weight corresponding to x_i , $i = 0, 1, \cdots N$,

and $\gamma_p = \sum_{i=0}^{N} T_p^2(x_i) \omega_j^c = \begin{cases} \pi, p = 0, \\ \dfrac{\pi}{2}, 1 \le p \le N. \end{cases}$ In addition, $\gamma_N = \dfrac{\pi}{2}$ if $\{x_i\}_{i=0}^N$ are the $N+1-$point Chebyshev Gauss, or Chebyshev Gauss-Radau points, $\gamma_N = \pi$ if $\{x_i\}_{i=0}^N$ are the $N+1-$point Chebyshev Gauss-Lobatto points.

4. Convergence Analysis

4.1. Some Spaces

For simplicity, we denote $\left(\dfrac{\partial^k v}{\partial_x k}\right)(x)$ by $\partial_x^k v(x)$, $0 \le k \le m$.

For non-negative integer m , we define $H_{\omega^{\alpha,\beta}}^m(-1,1) := \{ v := \partial_x^k v \in L_{\omega^{\alpha,\beta}}^2(-1,1), 0 \le k \le m \}$ with the norm as $\|v\|_{H_{\omega^{\alpha,\beta}}^m(-1,1)} := \left(\sum_{k=0}^{m} \|\partial_x^k v\|_{L_{\omega^{\alpha,\beta}}^2(-1,1)}^2 \right)^{\frac{1}{2}}$.

For a nonnegative integer N , we define the semi-norm

$$|v|_{H_{\omega^{\alpha,\beta}}^{m;N}(-1,1)} := \left(\sum_{k=\min(m,N+1)}^{m} \|\partial_x^k v\|_{L_{\omega^{\alpha,\beta}}^2(-1,1)}^2 \right)^{\frac{1}{2}},$$

when $\alpha = \beta = 0$, we denote $H_{\omega^{0,0}}^{m;N}(-1,1)$ by $H^{m;N}(-1,1)$. When $\alpha = \beta = -\dfrac{1}{2}$, we denote ω^c by $\omega^{-\frac{1}{2}}$.

The space $L^\infty(\text{-}1,1)$ is the Banach space of the measurable functions $u : (-1,1) \to R$ that is bounded outside a set of measure zero, equipped the norm $\|u\|_{L^\infty(\text{-}1,1)} := \underset{x\in(-1,1)}{ess \sup} |u(x)|$. P_N is the space of all polynomials of degree not exceeding N .

4.2. Lemmas

Lemma 1. [6,7] If $u \in H_{\omega^c}^m(-1,1)$, $m \ge 1$, $I_N v := \sum_{i=0}^{N} v(x_i) F_i(x)$, $v \in C([-1,1])$, then

$$\|u - I_N u\|_{L_{\omega^c}^2(\text{-}1,1)} \le CN^{-m} |u|_{H_{\omega^c}^m(\text{-}1,1)}, \tag{9}$$

$$\left\|u-I_N u\right\|_{L^\infty(-1,1)} \le CN^{\frac{1}{2}-m}\left|u\right|_{H^m_{\omega^c}(-1,1)}, \tag{10}$$

where I_N is the interpolation operater associated with the $N+1-$point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points $\{x_i\}_{i=0}^N$.

Lemma 2. [6,7] If $u\in H^m_{\omega^c}(-1,1)$, $v\in H^m_{\omega^c}(-1,1)$, $m\ge 1$ and $\Psi\in P_N$, then there exists a constant C independent of N such that

$$\left|\int_{-1}^{1} u(x)\Psi(x)\omega^c(x)dx-\sum_{j=0}^{N} u(x_j)\Psi(x_j)\omega_j^c\right| \le CN^{-m}\left|u\right|_{H^m_{\omega^c}(-1,1)}\left\|\Psi\right\|_{L^2_{\omega^c}(-1,1)}, \text{ and}$$

$$\left|\int_{-1}^{1} v(x)\Psi(x)dx-\sum_{j=0}^{N} v(x_j)\Psi(x_j)\omega_j\right| \le CN^{-m}\left|v\right|_{H^m_{\omega^c}(-1,1)}\left\|\Psi\right\|_{L^2(-1,1)}.$$

where x_j is the $N+1-$point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto point, corresponding weight ω_j^c, $j=0,1,\cdots,N$ and z_j is $N+1$ -point Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto point, corresponding weight ω^c, $j=0,1,\cdots,N$.

Lemma 3. If $F_j(x)$, $j=0,1,\cdots,N$ are the $j-th$ Lagrange interpolation polynomials associated with the $N+1-$point Chebyshev Gauss, or Chebyshev Gauss-Radau, or Chebyshev Gauss-Lobatto points , then

$$\left\|I_N\right\|_{L^\infty(-1,1)} := \max_{x\in(-1,1)}\sum_{j=0}^{N}\left|F_j(x)\right| = O(\log N).$$

Lemma 4. [1,8] (Gronwall inequality) Suppose $u(x)$ $(-1\le x\le 1)$ is a non-negative, locally integrable function satisfying $u(x)\le v(x)+L\int_{-1}^{x}u(\tau)d\tau,$ where $L\ge 0$ is a constant, $v(x)$ is a integrable function, then there exists a constant C such that

$$u(x)\le v(x)+C\int_{-1}^{x}v(\tau)d\tau, \quad \left\|u(x)\right\|_{L^\infty(-1,1)}\le C\left\|v(x)\right\|_{L^\infty(-1,1)}.$$

Lemma 5. Assume that $e(x)$ is a non-negative integrable function and satisfies

$$e(x)\le v(x)+M_1\int_{-1}^{x}e(t)dt+M_2\int_{-1}^{\frac{qT}{2}(1+x)^2-1}e(\theta)d\theta, \quad (0\le M_1, M_2<+\infty),$$

where $v(x)$ is also a non-negative integrable function, then

$$e(x)\le v(x)+C\int_{-1}^{x}v(\tau)d\tau, \quad \left\|e(x)\right\|_{L^\infty(-1,1)}\le C\left\|v(x)\right\|_{L^\infty(-1,1)}.$$

Proof. Since

$$q\tau^2=\frac{qT}{2}(1+x)^2-1<x,$$

Then

$$e(x)\le v(x)+(M_1+M_2)\int_{-1}^{x}e(\theta)d\theta.$$

Lemma 6. [9] For all measurable function $f\ge 0$, the following generalized Hardy's inequality $\left(\int_a^b\left|Tf(x)\right|^q\omega_1(x)dx\right)^{\frac{1}{q}}\le C\left(\int_a^b\left|f(x)\right|^p\omega_2(x)dx\right)^{\frac{1}{p}}$ holds if and only if

$$\sup_{a<x<b}\left(\int_x^b\omega_1(t)dt\right)^{\frac{1}{q}}\left(\int_a^x\omega_2(t)^{1-p'}dt\right)^{\frac{1}{p'}}<\infty,\quad p'=\frac{p}{p-1}, 1<p\le q<\infty$$

where T is an operator of the form $(Tf)(x)=\int_a^x k(x,t)f(t)dt$.

Lemma 7. [10] For every bounded function $v(x)$, there exists a constant C independent of v such that $\sup_N\left\|I_N v\right\|_{L^2_{\omega^c}(-1,1)}\le C\left\|v\right\|_{L^\infty(-1,1)}$.

4.3. Theorems

4.3.1. Convergence Analysis in $L^\infty(-1,1)$ Space

Theorem 1. Suppose $u(x)$ is the exact solution to (3) and $u^N(x)$ is the approximate solution obtained by using the spectral collocation schemes (6). Then for N sufficiently large, there is

$$\left\|u-u^N(x)\right\|_{L^\infty(-1,1)}\le CN^{\frac{1}{2}-m}\left(\left|u\right|_{H^m_{\omega^c}(-1,1)}+\left(K_1^*+K_2^*\right)\left\|u\right\|_{L^\infty(-1,1)}\right), \tag{11}$$

where $K_i^* := \max_{-1\le x\le 1}\left|K_i(x,\cdot)\right|_{H^m_{\omega^c}(-1,1)}$, $i=1,2.$

Proof. Subtracting (6) from (5) gives

$$u(x_i)-u_i=\int_{-1}^{1}K_1(x_i,\mathrm{z})u\left(s_{x_i}(z_k)\right)dz+\int_{-1}^{-1}K_2(x_i,\mathrm{v})u\left(t_{x_i}(v_k)\right)dv$$

$$-\int_{-1}^{1}K_1(x_i,\mathrm{z})u^N\left(s_{x_i}(z_k)\right)dz-\int_{-1}^{-1}K_2(x_i,v)u^N\left(t_{x_i}(v_k)\right)dv$$

$$+\int_{-1}^{1}K_1(x_i,\mathrm{z})u^N\left(s_{x_i}(z_k)\right)dz+\int_{-1}^{-1}K_2(x_i,v)u^N\left(t_{x_i}(v_k)\right)dv$$

$$-\sum_{j=0}^{N}u_j\sum_{k=0}^{N}\omega_k K_1(x_i,z_k)F_j\left(s_{x_i}(z_k)\right)-\sum_{j=0}^{N}u_j\sum_{k=0}^{N}\omega_k K_2(x_i,v_k)F_j\left(t_{x_i}(v_k)\right)$$

$$=\int_{-1}^{1}K_1(x_i,\mathrm{z})\left(u\left(s_{x_i}(z_k)\right)-u^N\left(s_{x_i}(z_k)\right)\right)dz+\int_{-1}^{-1}K_2(x_i,\mathrm{v})\left(u\left(t_{x_i}(v_k)\right)-u^N\left(t_{x_i}(v_k)\right)\right)dv$$

$$+\int_{-1}^{1}K_1(x_i,\mathrm{z})u^N\left(s_{x_i}(z_k)\right)dz-\sum_{k=0}^{N}K_1(x_i,\mathrm{z_k})u^N\left(s_{x_i}(z_k)\right)\omega_k$$

$$+\int_{-1}^{1}K_2(x_i,v)u^N\left(t_{x_i}(v_k)\right)dv-\sum_{k=0}^{N}K_2(x_i,\mathrm{v_k})u^N\left(t_{x_i}(v_k)\right)\omega_k$$

$$=\int_{-1}^{1}K_1(x_i,\mathrm{z})e\left(s_{x_i}(z)\right)dz+\int_{-1}^{1}K_2(x_i,v)e\left(t_{x_i}(v)\right)dv+J_1(x_i)+J_2(x_i),$$

Then

$$e(x_i)=\int_{-1}^{1}K_1(x_i,\mathrm{z})e\left(s_{x_i}(z)\right)dz+\int_{-1}^{1}K_2(x_i,v)e\left(t_{x_i}(v)\right)dv+J_1(x_i)+J_2(x_i),\ i=0,1,\cdots N, \tag{12}$$

where $e(x):=u(x)-u^N(x)$,

$$J_1(x):=\int_{-1}^{1}K_1(x,\mathrm{z})u^N\left(s_x(z)\right)dz-\sum_{k=0}^{N}K_1(x,\mathrm{z_k})u^N\left(s_x(z_k)\right)\omega_k,$$

$$J_2(x):=\int_{-1}^{1}K_2(x,\mathrm{v})u^N\left(t_x(v)\right)dz-\sum_{k=0}^{N}K_2(x,\mathrm{v_k})u^N\left(t_x(v_k)\right)\omega_k.$$

By Lemma 2,

$$\left|J_1(\mathrm{x})\right|\le CN^{-m}\left|K_1(x,\cdot)\right|_{H^{m;N}(-1,1)}\left\|u^N\left(s_x(\cdot)\right)\right\|_{L^2(-1,1)},\quad \left|J_2(\mathrm{x})\right|\le CN^{-m}\left|K_2(x,\cdot)\right|_{H^{m;N}(-1,1)}\left\|u^N\left(\mathrm{t}_x(\cdot)\right)\right\|_{L^2(-1,1)}. \tag{13}$$

Multiplying $F_j(x)$ on both sides of the error equation (12) and summing up from $i=0$ to $i=N$ yield

$$\sum_{i=0}^{N}F_i(x)e(x_i)=\sum_{i=0}^{N}F_i(x)\int_{-1}^{1}K_1(x_i,\mathrm{z})e\left(s_{x_i}(z)\right)dz+\sum_{i=0}^{N}F_i(x)\int_{-1}^{1}K_2(x_i,v)e\left(t_{x_i}(v)\right)dv+\sum_{i=0}^{N}F_i(x)J_1(x_i)+\sum_{i=0}^{N}F_i(x)J_2(x_i)$$

$$=I_N\int_{-1}^{1}K_1(x_i,\mathrm{z})e\left(s_{x_i}(z)\right)dz+I_N\int_{-1}^{1}K_2(x_i,v)e\left(t_{x_i}(v)\right)dv+I_NJ_1(x_i)+I_NJ_2(x_i),$$

besides,

$$\sum_{i=0}^{N}F_i(x)e(x_i)=\sum_{i=0}^{N}\left(u(x_i)-u_i\right)F_i(x)=\sum_{i=0}^{N}u(x_i)F_i(x)-\sum_{i=0}^{N}u_iF_i(x)=I_Nu-u^N,$$

So

$$I_Nu-u^N=I_N\int_{-1}^{1}K_1(x,\mathrm{z})e\left(s_x(z)\right)dz+I_N\int_{-1}^{1}K_2(x,v)e\left(t_x(v)\right)dv+I_NJ_1(x)+I_NJ_2(x). \tag{14}$$

Consequently,

$$\mathrm{e(x)} = u(x) - u^N(x) = I_N u(x) - u^N(x) + u(x) - I_N u(x)$$

$$= I_N \int_{-1}^{1} K_1(x,\mathrm{z}) e(s_x(z)) dz + I_N \int_{-1}^{1} K_2(x,v) e(t_x(v)) dv + I_N J_1(x) + I_N J_2(x) + u(x) - I_N u(x)$$

$$= \int_{-1}^{1} K_1(x,\mathrm{z}) e(s_x(z)) dz + \int_{-1}^{1} K_2(x,v) e(t_x(v)) dv + u(x) - I_N u(x) + I_N J_1(x) + I_N J_2(x) + J_3(x) + J_4(x) \tag{15}$$

where $J_0 := u - I_N u$,

$$J_3 := I_N \int_{-1}^{1} K_1(x,\mathrm{z}) e(s_x(z)) dz - \int_{-1}^{1} K_1(x,\mathrm{z}) e(s_x(z)) dz,\ J_4 := I_N \int_{-1}^{1} K_2(x,v) e(t_x(v)) dv - \int_{-1}^{1} K_2(x,\mathrm{v}) e(t_x(v)) dv.$$

By Lemma 5, we get

$$\|e\|_{L^\infty(-1,1)} \le C\left(\|J_0\|_{L^\infty(-1,1)} + \|I_N J_1\|_{L^\infty(-1,1)} + \|I_N J_2\|_{L^\infty(-1,1)} + \|J_3\|_{L^\infty(-1,1)} + \|J_4\|_{L^\infty(-1,1)}\right). \tag{16}$$

Using Lemma 1 for $J_0(x)$ yields

$$\|J_0(x)\|_{L^\infty(-1,1)} = \|u - I_N u\|_{L^\infty(-1,1)} \le CN^{\frac{1}{2}-m} |u|_{H_{\omega^c}^{m;N}(-1,1)}. \tag{17}$$

Using Lemma 7 and (13), we have

$$\max_{-1\le x\le 1} |J_1(x)| \le CN^{-m} K_1^* \max_{-1\le x\le 1} \|u^N(s_x(\cdot))\|_{L^2(-1,1)} \le CN^{-m} K_1^* \|u^N\|_{L^\infty(-1,1)} \le CN^{-m} K_1^* \left(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\right) \tag{18}$$

By Lemma 3,

$$\|I_N J_1(x)\|_{\infty(-1,1)} \le \|I_N\|_{L^\infty(-1,1)} \|J_1(x)\|_{\infty(-1,1)} \le CN^{-m} (\log N) K_1^* \left(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\right). \tag{19}$$

Similarly,

$$\|I_N J_2(x)\|_{\infty(-1,1)} \le CN^{-m} (\log N) K_2^* \left(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\right). \tag{20}$$

According to Lemma 1 with $m = 1$ to $J_3(x)$ yields

$$\begin{aligned}
\|J_3\|_{L^\infty(-1,1)} &\le CN^{-\frac{1}{2}} \left| \int_{-1}^{1} K_1(x,z) e(s_x(z)) dz \right|_{H_{\omega^c}^{1;N}(-1,1)} = CN^{-\frac{1}{2}} \left| \int_{-1}^{x} \tilde{K}_1(x,\mathrm{s}) e(s) ds \right|_{H_{\omega^c}^{1;N}(-1,1)} \\
&= CN^{-\frac{1}{2}} \left\| \tilde{K}_1(x,x) e(x) + \int_{-1}^{x} e(s) \frac{\partial}{\partial x} \tilde{K}_1(x,\mathrm{s}) ds \right\|_{L^2_{\omega^C}(-1,1)} \\
&\le CN^{-\frac{1}{2}} \left\| \tilde{K}_1(x,x) e(x) \right\|_{L^2_{\omega^C}(-1,1)} + CN^{-\frac{1}{2}} \left\| \int_{-1}^{x} e(s) \frac{\partial}{\partial x} \tilde{K}_1(x,\mathrm{s}) ds \right\|_{L^2_{\omega^C}(-1,1)} \\
&= CN^{-\frac{1}{2}} \|e(x)\|_{L^2_{\omega^C}(-1,1)} + CN^{-\frac{1}{2}} \left\| \int_{-1}^{-1} e(s) ds \right\|_{L^2_{\omega^C}(-1,1)} \\
&\le CN^{-\frac{1}{2}} \|e(x)\|_{L^\infty(-1,1)}
\end{aligned} \tag{21}$$

Similarly,

$$\|J_4\|_{L^\infty(-1,1)} = CN^{-\frac{1}{2}} \left| \int_{-1}^{\frac{qT}{2}(1+x)^2-1} \tilde{K}_2(x,\mathrm{t}) e(t) dt \right|_{H_{\omega^c}^{1;N}(-1,1)}.$$

$$= CN^{-\frac{1}{2}} \left\| \int_{-1}^{x} \tilde{K}_2\left(x, \frac{qT}{2}(1+u)^2 - 1\right) e\left(\frac{qT}{2}(1+u)^2 - 1\right)(qt(1+u))du \right\|_{H^{1;N}_{\omega^c}(-1,1)}$$

$$\le CN^{-\frac{1}{2}} \left\| \begin{array}{l} \tilde{K}_2\left(x, \frac{qT}{2}(1+x)^2 - 1\right) e\left(\frac{qT}{2}(1+x)^2 - 1\right)(qt(1+x)) \\ + \int_{-1}^{x} e\left(\frac{qT}{2}(1+x)^2 - 1\right)(qt(1+x)) \frac{\partial}{\partial x} \tilde{K}_2\left(x, \frac{qT}{2}(1+x)^2 - 1\right) dx \end{array} \right\|_{L^2_{\omega^C}(-1,1)} \le CN^{-\frac{1}{2}} \|e\|_{L^\infty(-1,1)}. \quad (22)$$

So

$$\|e\|_{L^\infty(-1,1)}$$

$$\le CN^{\frac{1}{2}-m} |u|_{H^{m;N}_{\omega^c}(-1,1)} + CN^{-m} (\log N)\left(K_1^* + K_2^*\right)\|u\|_{L^\infty(-1,1)} \quad (23)$$

Since $\lim_{N\to\infty} \frac{N^{-m}\log^N}{N^{\frac{1}{2}-m}} = \lim_{N\to\infty} \frac{\log^N}{N^{\frac{1}{2}}} = 0,$ then, for N sufficiently large, $N^{-m}\log^N < N^{\frac{1}{2}-m}$.

Hence

$$\|e\|_{L^\infty(-1,1)} \le CN^{\frac{1}{2}-m}\left(|u|_{H^{m;N}_{\omega^c}(-1,1)} + \left(K_1^* + K_2^*\right)\|u\|_{L^\infty(-1,1)}\right).$$

This completes the proof of this theorem.

4.3.2. Convergence Analysis in $L^2_{\omega^C}(-1,1)$ Space

Theorem 2. Suppose $u(x)$ is the exact solution to (3), and $u^N(x)$ is the approximate solution obtained by using the spectral collocation schemes (6). Then for N sufficiently large, we have

$$\left\|u(x) - u^N(x)\right\|_{L^2_{\omega^C}(-1,1)} \le CN^{-m}\left(K_1^* + K_2^* + 1\right)\left(\|u'\|_{L^2_{\omega^C}(-1,1)} + \|u\|_{L^\infty(-1,1)} + |u|_{H^{m;N}_{\omega^c}}\right),$$

where $K_i^* := \max_{-1\le x\le 1} |K_i(x,\cdot)|_{H^m_{\omega^c}(-1,1)}$, $i = 1,2$.

Proof. Applying Lemma 5, it follows from (15) that

$$e(x) \le C\int_{-1}^{x} \left|J_0(t) + I_N J_1(t) + I_N J_2(t) + J_3(t) + J_4(t)\right| dt + \left|J_0(x) + I_N J_1(x) + I_N J_2(x) + J_3(x) + J_4(x)\right|.$$

By Lemma 6, we have

$$\|e\|_{L^2_{\omega^C}(-1,1)} \le C\left(\|J_0\|_{L^2_{\omega^C}(-1,1)} + \|I_N J_1\|_{L^2_{\omega^C}(-1,1)} + \|I_N J_2\|_{L^2_{\omega^C}(-1,1)} + \|J_3\|_{L^2_{\omega^C}(-1,1)} + \|J_4\|_{L^2_{\omega^C}(-1,1)}\right).$$

Using Lemma 1 for $J_0(x)$ yields

$$\|J_0(x)\|_{L^2_{\omega^C}(-1,1)} = \|u - I_N u\|_{L^2_{\omega^C}(-1,1)} \le CN^{-m}|u|_{H^{m;N}_{\omega^c}(-1,1)}. \quad (24)$$

With Lemma 7, we get

$$\|I_N J_1(x)\|_{L^2_{\omega^C}(-1,1)} \le C\max_{-1\le x\le 1}|J_1(x)| \le CN^{-m}K_1^*\left(\|e\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\right). \quad (25)$$

From Theorem 1, let $m = 1$, then (11) becomes

$$\|e\|_{L^\infty(-1,1)} \le C(\|u'\|_{L^2_{\omega^C}(-1,1)} + \left(K_1^* + K_2^*\right)\|u\|_{L^\infty(-1,1)}),$$

This makes (25) become

$$\|I_N J_1(x)\|_{L^2_{\omega^C}(-1,1)} \le CN^{-m}K_1^*\left(\|u'\|_{L^2_{\omega^C}(-1,1)} + \left(K_1^* + K_2^* + 1\right)\|u\|_{L^\infty(-1,1)}\right). \quad (26)$$

Similarly,

$$\left\| I_N J_2(x) \right\|_{L^2_{\omega^C}(-1,1)} \le CN^{-m} K_2^* \left(\left\| u' \right\|_{L^2_{\omega^C}(-1,1)} + \left(K_1^* + K_2^* + 1 \right) \left\| \mathrm{u} \right\|_{L^\infty(-1,1)} \right). \tag{27}$$

As the same analysis in (21), we obtain that $\left\| J_3(x) \right\|_{L^2_{\omega^C}(-1,1)} \le CN^{-1} \left\| \mathrm{e} \right\|_{L^\infty(-1,1)}$.

From Theorem 1, we get

$$\left\| J_3(x) \right\|_{L^2_{\omega^C}(-1,1)} \le CN^{-\frac{1}{2}-m} K_2^* \left(\left| \mathrm{u} \right|_{H^{m;N}_{\omega^c}(-1,1)} + \left(K_1^* + K_2^* \right) \left\| \mathrm{u} \right\|_{L^\infty(-1,1)} \right). \tag{28}$$

Similarly,

$$\left\| J_4(x) \right\|_{L^2_{\omega^C}(-1,1)} \le CN^{-\frac{1}{2}-m} K_2^* \left(\left| \mathrm{u} \right|_{H^{m;N}_{\omega^c}(-1,1)} + \left(K_1^* + K_2^* \right) \left\| \mathrm{u} \right\|_{L^\infty(-1,1)} \right). \tag{29}$$

Hence, we have $\left\| e \right\|_{L^2_{\omega^C}(-1,1)} \le CN^{-m} \left(K_1^* + K_2^* + 1 \right) \left(\left| u \right|_{H^{m;N}_{\omega^c}(-1,1)} + \left\| u' \right\|_{L^2_{\omega^C}(-1,1)} + \left\| u \right\|_{L^\infty(-1,1)} \right)$.

This completes the proof of this Theorem.

5. Examples

5.1. Example 1

From (1), let $\tilde{K}_1(x,s) = e^{xs}, \tilde{K}_2(x,t) \equiv 0,\ g(x) = e^{2x} + \frac{1}{x+2}\left(e^{-(x+2)} - e^{x(x+2)} \right)$.

The corresponding exact solution is $u(x) = e^{2x},\ x \in [-1,1]$.

We use the numerical scheme (6).Numerical errors versus several values of N are displayed in Table 1 and Figure ①. These results indicate that the desired spectral accuracy is obtained. Figure ② presents the approximate solution ($N = 24$) and the exact solution, which are found in excellent agreement.

Table 1. *The errors $u - u^N$ versus the number of collocation points in L^∞ and $L^2_{\omega^C}$ norms.*

N	6	8	10	12	14
$L^\infty - error$	2.4058e-004	2.0357e-006	1.3388e-008	6.6231e-011	2.5646e-013
$L^2_{\omega^C} - error$	2.6350e-004	2.3946e-006	1.5653e-008	7.9650e-011	3.1722e-013
N	16	18	20	22	24
$L^\infty - error$	2.1316e-014	7.9936e-015	7.9936e-015	7.1054e-015	1.1546e-014
$L^2_{\omega^C} - error$	1.3511e-014	1.0471e-014	1.4066e-014	1.2207e-014	1.6227e-014

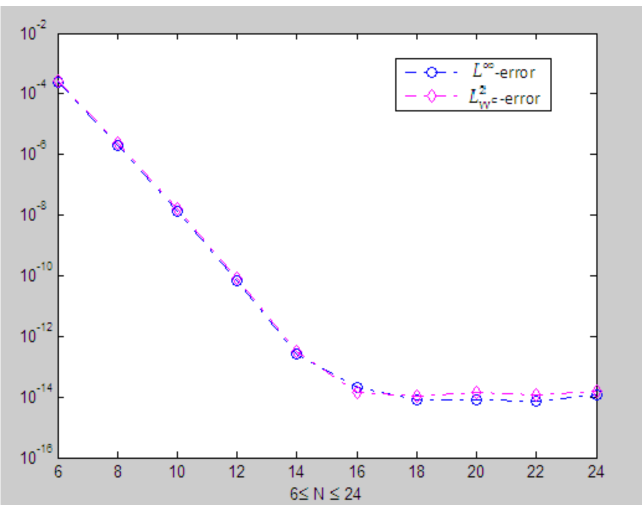

Figure 1. *The errors $u - u^N$ versus the number of collocation points in L^∞ and $L^2_{\omega^C}$ norms.*

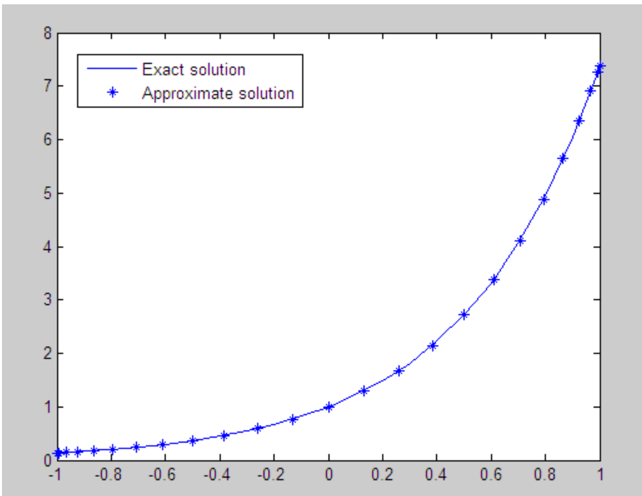

Figure 2. *Comparison between approximate solution u^N and the exact solution u.*

5.2. Example 2

From (1), let

$$R_1(\tau,\xi)=-(\tau+\xi),\ R_2(\tau,\eta)=-(\tau-\eta),\ T=2,\ q\tau=\frac{1}{2}\tau^2,\ f(\tau)=-2\tau\cos\tau+2\tau+2\sin\tau+\left(\frac{1}{2}\tau^2-\tau\right)\cos(\frac{1}{2}\tau^2)-\sin(\frac{1}{2}\tau^2).$$

The corresponding exact solution is $y(\tau)=\sin\tau,\ \tau\in[0,2]$.

Figure ③ plots the errors for $2\le N\le 20$ in both L^∞ and $L^2_{\omega^c}$ norms. The approximate solution ($N=20$) and the exact solution are displayed in Table 2. As expected, the errors decay exponentially which confirmed our theoretical predictions.

Table 2. *The errors versus the number of collocation points in L^∞ and $L^2_{\omega^c}$ norms.*

N	2	4	6	8	10
$L^\infty-error$	2.9023e-002	2.3295e-004	9.6846e-007	2.6571e-009	5.8210e-012
$L^2_{\omega^c}-error$	3.5380e-002	2.6275e-004	1.0136e-006	2.8130e-009	5.8353e-012
N	12	14	16	18	20
$L^\infty-error$	8.9928e-015	3.3307e-015	2.5535e-015	2.2204e-015	2.1094e-015
$L^2_{\omega^c}-error$	8.8042e-015	2.1409e-015	1.7146e-015	2.4297e-015	2.4104e-015

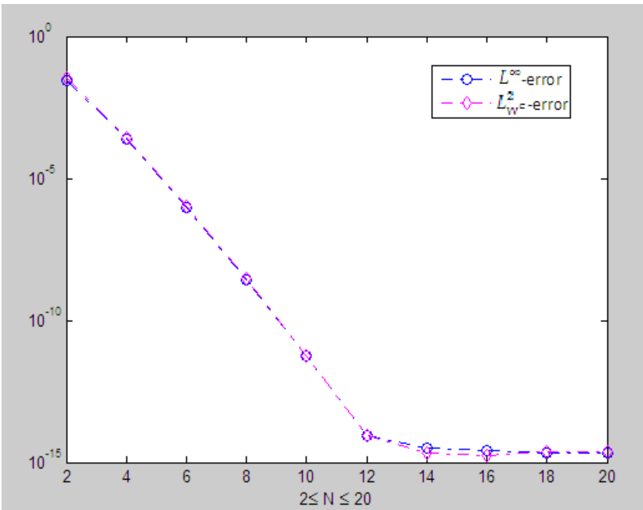

Figure 3. *Plots the errors for $2\le N\le 20$ in both L^∞ and $L^2_{\omega^c}$ norms.*

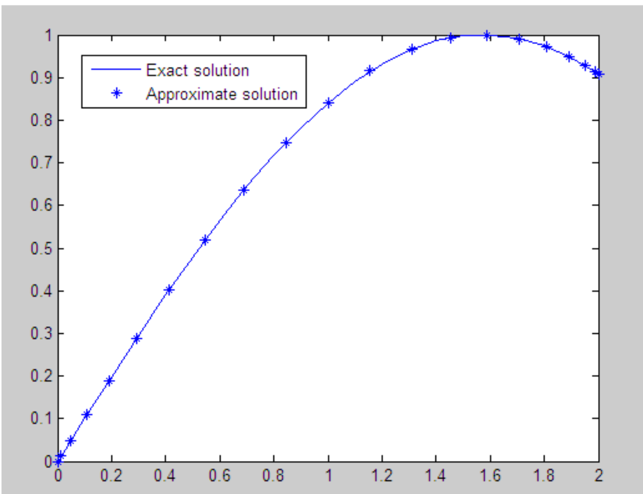

Figure 4. *Comparison between approximate solution ($N=20$) and the exact solution.*

6. Conclusion

We successfully solve the Volterra integral equation with vanishing delay by Chebyshev spectral-collocation method and provide a rigorous error analysis for this method. We get the conclusion that the error of the approximate solution decay exponentially in L^∞ norm and $L^2_{\omega^c}$ norm. We also carry out the numerical experiment which confirm the theoretical predicition of the exponential rate of convergence.

References

[1] T. Tang, X. Xu, J. Cheng, On spectral methods for Volterra type integral equations and the convergence analysis, J. Comput. Math. 26 (2008) 825-837.

[2] Y. Chen, and T. Tang, Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equation with a weakly singular kernel, Math. Comput. 79(2010), pp. 147–167.

[3] Z. Wan, Y. Chen, and Y. Huang, Legendre spectral Galerkin method for second-kind Volterraintegral equations, Front. Math. China, 4(2009), pp. 181–193.

[4] Z. Xie, X. Li and T. Tang, Convergence Analysis of Spectral Galerkin Methods for Volterra Type Integral Equations, J. Sci. Comput, 2012.

[5] Z. Gu and Y. Chen, Chebyshev spectral collocation method for Volterra integral equations [D·Master's Thesis], Contemporary Mathematics, Volume 586, 2013, pp. 163-170.

[6] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, Spectral method fundamentalsin single domains, Spring-Verlag, 2006.

[7] J. Shen, and T. Tang, Spectral and high-order methods with applications, Science Press, Beijing, 2006.

[8] D. Henry, Geometric theory of semilinear parabolic equations, Springer-Verlag, 1989.

[9] A. Kufner, and L. E. Persson, Weighted inequality of Hardy's Type, World scientific, NewYork, 2003.

[10] P. Nevai, Mean convergence of Lagrange interpolation, III, Trans. Amer. Math. Soc., 282(1984), pp. 669–698.