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Darboux transformation of lax pair for an integrable coupling of the integrable differential-difference equation

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Abstract: An integrable coupling of the known integrable differential-difference equation and its Lax pair are presented. Based on the gauge transformation between the corresponding four-by- four matrix spectral problems, a Darboux transformation of Lax pair for the integrable coupling is established. As an application of the obtained Darboux transformation, an explicit solution is given.

Keywords: Integrable Differential-Difference Equation, Integrable Coupling, Darboux, Transformation, Explicit Solution

1. Introduction

Since the original work of Fermi, Pasta and Ulam in the 1960s [1], integrable differential- difference equations has attracted wide interest, and has been applied in many fields of Physics. Many integrable differential-difference equations have been deduced. Their integrable properties have been studied from different points of view. Such as the inverse scattering transformation [2], the symmetries and master symmetries[3], Hamiltonian structure and bi-Hamiltonian structure, [4-6], constructing complexiton solutions by the Casorati determinant[7], the symmetry constraints [8], and so forth.

Recently, the investigation of integrable coupling system of soliton equations has attracted much attention. The integrable couplings originate from the work on perturbations around solutions of evolution equations [9], and the perturbation bundle [10]. A few approaches to obtain integrable coupling systems of the integrable evolution equations are proposed. For example, perturbation methods [11-12], enlarging spectral problems [13], semi-direct sums of Lie algebras [14-17], and so on.

For a given integrable differential-difference equation

$$y_{nt} = K_n(y_n), \tag{1}$$

In which $y_n = y(n,t)$ is a vector-valued real function defined over $Z \times R$. We actually want to derive a bigger, triangular integrable differential-difference equation as follows:

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix}_t = \begin{pmatrix} K_n(y_n) \\ \xi(y_n, z_n) \end{pmatrix} \tag{2}$$

Here z_n is a new vector-valued real function defined over $Z \times R$, and the vector-valued function $\xi(y_n, z_n)$ should satisfy the non-triviality condition $\frac{\partial \xi(y_n, z_n)}{\partial [y_n]} \neq 0$, where $[y_n] = (y_n . y_{n+1}, y_{n-1}, \ldots)$.This statement means that the other differential-difference equation in the bigger system (2) involves the dependent variables of the original equation (1). The Eq. (2) is called an integrable coupling system of Eq. (1).

In addition, Darboux transformation is a powerful tool to obtain explicit solutions of integrable differential-difference equations [18-24]. The discrete matrix spectral problem of integrable differential-difference equation plays a key role in the theory of Darboux transformation. From the associated discrete matrix spectral problem, we may construct the Darboux transformation of integrable differential-difference equations and obtain explicit solutions of integrable differential-difference equations.

In Ref [21], an integrable differential-difference equation

$$\begin{cases} r_{n,t} = r_{n+1} - r_n^2 s_n, \\ s_{n,t} = r_n s_n^2 - s_{n-1}, \end{cases} \tag{3}$$

and its integrate family are derived from a matrix spectral problem. Hamiltonian structure of this integrable family is established by the discrete trace identity .Its related Darboux transformation is discussed. If we set the equation (3) in following form

$$y_{n,t} = K_n(y_n),$$

where $y_n = (r_n, s_n)^T$, then a first-order perturbation system of equations (3) may be represented as

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix}_t = \begin{pmatrix} K_n(y_n) \\ K_n(y_n) + K_n^{'}(y_n)[z_n] \end{pmatrix} \tag{4}$$

In which $z_n = (u_n, w_n)^T$, $K_n^{'}(y_n)[z_n]$ denotes the Gateaux derivative of $K_n(y_n)$ with respect to y_n in a direction z_n .It is easy to obtain that the above first-order perturbation system (4) of the equations (3) become

$$\begin{cases} r_{n,t} = r_{n+1} - r_n^2 s_n, \\ s_{n,t} = r_n s_n^2 - s_{n-1}, \\ u_{n,t} = -r_{n+1} + u_{n+1} + r_n^2 s_{n-1} + r_n^2 w_{n-1} + 2r_n s_{n-1} u_n, \\ w_{n,t} = s_{n-1} - w_{n-1} - r_{n+1}^2 s_n + s_n^2 u_{n-1} + 2r_{n-1} s_n w_n. \end{cases} \tag{5}$$

Because first two equations in the Eq.(5) form the Eq.(3),the Eq.(5) is an integrable coupling of the integrable differential-difference equation (3).Here, let $f(n)$ be a lattice function, the shift operator E, the inverse of E are defined by

$$E(f(n)) = f(n+1), \qquad E^{-1}(f(n)) = f(n-1), \quad n \in Z.$$

By a direct calculation, we can obtain that the integrable coupling (5) has following Lax pair

$$E\phi_n = U_n \phi_n = \begin{pmatrix} \lambda & \lambda r_n & 0 & \lambda u_n \\ s_n & 1 + r_n s_n & w_n & r_n w_n + s_n u_n \\ 0 & 0 & \lambda & \lambda r_n \\ 0 & 0 & s_n & 1 + r_n s_n \end{pmatrix} \phi_n,$$

$$\chi_n = \begin{pmatrix} r_n \\ s_n \\ u_n \\ w_n \end{pmatrix}, \qquad \phi_n = \begin{pmatrix} \phi_n^1 \\ \phi_n^2 \\ \phi_n^3 \\ \phi_n^4 \end{pmatrix} \tag{6}$$

and

$$\varphi_{n,t} = V_n \varphi_n,$$

in which

$$v_n = \begin{pmatrix} \frac{\lambda}{2} - r_n s_{n-1} & \lambda r_n & \frac{\lambda}{2} - r_n s_{n-1} - r_n w_{n-1} - s_{n-1} u_n & -r_n \lambda + u_n \lambda \\ s_{n-1} & -\frac{\lambda}{2} + r_n s_{n-1} & -s_{n-1} + w_{n-1} & -\frac{\lambda}{2} + r_n s_{n-1} + r_n w_{n-1} + s_{n-1} u_n \\ 0 & 0 & \frac{\lambda}{2} - r_n s_{n-1} & \lambda r_n \\ 0 & 0 & s_{n-1} & -\frac{\lambda}{2} + r_n s_{n-1} \end{pmatrix} \tag{7}$$

Namely, the compatibility conditions of the Eq. (6) and Eq. (7) is

$$U_{n,t} = (EV_n)U_n - U_n V_n. \tag{8}$$

and the Eq. (8) yields the Eq. (5).

This paper is organized as follows. In section 2,by means of the gauge transformation of the Lax pair, we construct a Darboux transformation of Lax pair of the Eq.(5).To the best of our knowledge, Darboux transformation of Lax pair of this triangular integrable coupling has not been studied. In Section 3, as an application of Darboux transformation, an explicit solution of the Eq. (5) is deduced. Some conclusions and remarks are given in the final Section.

2. Darboux Transformation

It is well known that Darboux transformations provide us with a purely algebraic, powerful method to find explicit solutions of some integrable differential-difference equations (or discrete integrable systems). A matrix spectral problem of an integrable differential-difference equation plays a key role [19-24]. we know that a gauge transformation of matrix spectral problems is called a Darboux transformation if it transforms the spectral problem into another spectral problem of the same type. In what follows, we are going to establish a Darboux transformation of the integrable differential-difference equation (5).In theory of Darboux transformation, a key problem is that the transformation matrix is properly presented. We introduce gauge transformation

$$\widetilde{\varphi}_n = \Pi_n^{(N)} \varphi_n \tag{9}$$

Here, we assume $\Pi_n^{(N)}$ is of the form

$$\begin{pmatrix} \lambda^N+\sum_{i=0}^{N-1}a_n^{(i)}\lambda^i & \sum_{i=0}^{N-1}b_n^{(i)}\lambda^{i+1} & \lambda^N+\sum_{i=0}^{N-1}f_n^{(i)}\lambda^i & \sum_{i=0}^{N-1}g_n^{(i)}\lambda^{i+1} \\ \sum_{i=0}^{N-1}c_n^{(i)}\lambda^i & \lambda^N+\sum_{i=0}^{N-1}d_n^{(i)}\lambda^i & \sum_{i=0}^{N-1}h_n^{(i)}\lambda^i & \lambda^N+\sum_{i=0}^{N-1}l_n^{(i)}\lambda^i \\ 0 & 0 & \lambda^N+\sum_{i=0}^{N-1}a_n^{(i)}\lambda^i & \sum_{i=0}^{N-1}b_n^{(i)}\lambda^{i+1} \\ 0 & 0 & \sum_{i=0}^{N-1}c_n^{(i)}\lambda^i & \lambda^N+\sum_{i=0}^{N-1}d_n^{(i)}\lambda^i \end{pmatrix} \tag{10}$$

in which N is a natural number, $a_n^{(i)},b_n^{(i)},c_n^{(i)},d_n^{(i)},f_n^{(i)},g_n^{(i)},h_n^{(i)},l_n^{(i)},i=1,\cdots N-1$, are undetermined functions of variables n and t. The Eq. (9) can transform two spectral problems (7) and (8) into

$$E\tilde{\phi}_n=\tilde{\phi}_{n+1}=\tilde{U}_n\tilde{\phi}_n. \tag{11}$$

$$\widetilde{\varphi}_{n,t}=\widetilde{V}_n\widetilde{\varphi}_n. \tag{12}$$

$$\widetilde{U}_n=\Pi_{n+1}^{(N)}U_n(\Pi_n^{(N)})^{-1},\widetilde{V}_n=(\Pi_n^{(N)})_t+\Pi_n^{(N)}V_n(\Pi_n^{(N)})^{-1}. \tag{13}$$

In what follows, we determine $\Pi_n^{(N)}$ such that $\widetilde{U}_n$ and $\widetilde{V}_n$ in Eq. (13) have the same form with U_n and V_n in Eq. (6) and Eq. (7) respectively. Let

$$\phi_n=(\phi_n^1(t,\lambda),\phi_n^2(t,\lambda),\phi_n^3(t,\lambda),\phi_n^4(t,\lambda))^T,$$
$$\psi_n=(\psi_n^1(t,\lambda),\psi_n^2(t,\lambda),\psi_n^4(t,\lambda))^T$$ be

two real linear independent solutions of Eq. (7) and Eq. (8), and use them to define following linear algebraic systems for $a_n^{(i)},b_n^{(i)},c_n^{(i)},d_n^{(i)},f_n^{(i)},g_n^{(i)},h_n^{(i)}$ and $l_n^{(i)},i=1,2,3,N-1$.

$$\sum_{i=0}^{N-1}(a_n^{(i)}+\alpha_j[n]b_n^{(i)}\lambda_j+\beta_j[n]f_n^{(i)}+\gamma_j[n]g_n^{(i)}\lambda_j)\lambda_j^i=-(1+\beta_j[n])\lambda_j^N,1\le j\le 2N. \tag{14}$$

$$\sum_{i=0}^{N-1}(c_n^{(i)}+\alpha_j[n]d_n^{(i)}\lambda_j+\beta_j[n]f_n^{(i)}+\gamma_j[n]l_n^{(i)}\lambda_j)\lambda_j^i=-(\alpha_j[n]+\gamma_j[n])\lambda_j^N,1\le j\le 2N. \tag{15}$$

$$\sum_{i=0}^{N-1}(\beta_j[n]a_n^{(i)}+\gamma_j[n]b_n^{(i)}\lambda_j)\lambda_j^i=-\beta_j[n]\lambda_j^N,1\le j\le 2N. \tag{16}$$

$$\sum_{i=0}^{N-1}(\beta_j[n]c_n^{(i)}+\gamma_j[n]d_n^{(i)}\lambda_j)\lambda_j^i=-\gamma_j[n]\lambda_j^N,1\le j\le 2N. \tag{17}$$

In which

$$\begin{cases}\alpha_j[n]=\dfrac{\phi_n^2(t,\lambda_j)-\kappa_j\psi_n^2(t,\lambda_j)}{\phi_n^1(t,\lambda_j)-\kappa_j\psi 1(t,\lambda_j)},\\ \beta_j[n]=\dfrac{\phi_n^3(t,\lambda_j)-\kappa_j\psi_n^3(t,\lambda_j)}{\phi_n^1(t,\lambda_j)-\kappa_j\psi 1(t,\lambda_j)}, \quad j=1,2,3,\cdots 2N.\\ \gamma_j[n]=\dfrac{\phi_n^4(t,\lambda_j)-\kappa_j\psi_n^4(t,\lambda_j)}{\phi_n^1(t,\lambda_j)-\kappa_j\psi 1(t,\lambda_j)},\end{cases} \tag{18}$$

$\lambda_j,\kappa_j,j=1,2,3\cdots 2N$, are suitably chosen such that all the determinants of coefficients for $Eq.(14)\sim Eq.(18)$ are nonzero. Therefore, $a_n^{(i)},b_n^{(i)},c_n^{(i)},d_n^{(i)},f_n^{(i)},g_n^{(i)},h_n^{(i)},l_n^{(i)},i=1,2,\quad 3,\cdots N-1$, are uniquely determined. From $Eq.(14)\sim Eq.(17)$, it is easy to get that $\det(\Pi_n^{(N)})$ is $4N$ th-order polynomial of $\lambda_j,j=1,2,\cdots 2N$, are all its roots.

Thus, we have

$$\det(\Pi_n^{(N)})=\prod_{j=1}^{2N}(\lambda-\lambda_j)^2. \tag{19}$$

Proposition 1. The matrix $\widetilde{U}_n$ defined by Eq. (13) has the same form as U_n, in which the old potentials r_n,s_n,u_n and w_n are mapped into new potentials $\widetilde{r}_n,\widetilde{s}_n,\widetilde{u}_n$ and $\widetilde{w}_n$ according to

$$\begin{cases}\widetilde{r}_n=r_n-b_n^{(N-1)},\\ \widetilde{s}_n=s_n+c_{n+1}^{(N-1)},\\ \widetilde{u}_n=u_n+b_n^{(N-1)}-g_n^{(N-1)},\\ \widetilde{w}_n=w_n-c_{n+1}^{(N-1)}+h_{n+1}^{(N-1)}.\end{cases} \tag{20}$$

Proof: Let

$$a_n=\lambda^N+\sum_{i=0}^{N-1}a_n^{(i)}\lambda^i,b_n=\sum_{i=0}^{N-1}b_n^{(i)}\lambda^{i+1},c_n=\sum_{i=0}^{N-1}c_n^{(i)}\lambda^i,d_n=\lambda^N+\sum_{i=0}^{N-1}d_n^{(i)}\lambda^i,$$

$$f_n=\lambda^N+\sum_{i=0}^{N-1}f_n^{(i)}\lambda^i,g_n=\sum_{i=0}^{N-1}g_n^{(i)}\lambda^{i+1},h_n=\sum_{i=0}^{N-1}h_n^{(i)}\lambda^i,l_n=\lambda^N+\sum_{i=0}^{N-1}l_n^{(i)}\lambda^i.$$

It is easy to obtain

$$\Pi_{n+1}^{(N)}U_n(\Pi_n^{(N)})^*=\begin{pmatrix}\Gamma_{11}(\lambda,n) & \Gamma_{12}(\lambda,n) & \Gamma_{13}(\lambda,n) & \Gamma_{14}(\lambda,n)\\ \Gamma_{21}(\lambda,n) & \Gamma_{22}(\lambda,n) & \Gamma_{23}(\lambda,n) & \Gamma_{24}(\lambda,n)\\ 0 & 0 & \Gamma_{11}(\lambda,n) & \Gamma_{12}(\lambda,n)\\ 0 & 0 & \Gamma_{21}(\lambda,n) & \Gamma_{22}(\lambda,n)\end{pmatrix}. \quad (21)$$

We can find that $\Gamma_{11}(\lambda,n),\Gamma_{12}(\lambda,n),\Gamma_{13}(\lambda,n)$ and $\Gamma_{14}(\lambda,n)$ are $(4N+1)$ order polynomials in λ , $\Gamma_{21}(\lambda,n),\Gamma_{22}(\lambda,n),\Gamma_{23}(\lambda,n)$ and $\Gamma_{24}(\lambda,n)$ are $4N$ order polynomials in λ.Furthermore, from Eq. (6) and Eq. (7), we get

$$\alpha_j[n+1]=\frac{\mu_j(n)}{\upsilon_j(n)},\quad \beta_j[n+1]=\frac{\sigma_j(n)}{\upsilon_j(n)},\quad \gamma_j[n+1]=\frac{\omega_j(n)}{\upsilon_j(n)},\quad j=1,2,\cdots 2N, \quad (22)$$

With

$$\begin{cases}\mu_j(n)=s_n+(1+r_ns_n)\alpha_j[n]+w_n\beta_j[n]+(s_nu_n+r_nw_n)\gamma_j[n],\\ \upsilon_j(n)=\lambda_j+r_n\lambda_j\alpha_j[n]+u_n\lambda_j\gamma_j[n],\\ \sigma_j(n)=\lambda_j\beta_j[n]+r_n\lambda_j\gamma_j[n],\\ \omega_j(n)=s_n\beta_j[n]+(1+r_ns_n)\gamma_j[n].\end{cases}\quad j=1,2,\cdots 2N. \quad (23)$$

According to Eq. (22) and Eq. (23), we can get the expressions of $a_{n+1},b_{n+1},c_{n+1},d_{n+1}$, $f_{n+1},g_{n+1},h_{n+1},l_{n+1}$.Making use of $Eq.(14)\sim Eq.(18)$, through a tedious but direct calculation, we can obtain

$$\Gamma_{ij}(\lambda_k,n)=0,i=1,2,j=1,2,3,4,k=1,2,\cdots 2N.$$

Therefore, we have

$$\Pi_{n+1}^{(N)}U_n(\Pi_n^{(N)})^*=\det(\Pi_n^{(N)})M_n,$$

with

$$M_n=\begin{pmatrix}m_{11}^{(1)}\lambda+m_{11}^{(0)} & m_{12}^{(1)}\lambda+m_{12}^{(0)} & m_{13}^{(1)}\lambda+m_{13}^{(0)} & m_{14}^{(1)}\lambda+m_{14}^{(0)}\\ m_{21}^{(0)} & m_{22}^{(0)} & m_{23}^{(0)} & m_{24}^{(0)}\\ 0 & 0 & m_{11}^{(1)}\lambda+m_{11}^{(0)} & m_{12}^{(1)}\lambda+m_{12}^{(0)}\\ 0 & 0 & m_{21}^{(0)} & m_{22}^{(0)}\end{pmatrix},$$

Where $m_{11}^{(1)},m_{12}^{(1)},m_{13}^{(1)},m_{14}^{(1)}$ and $m_{ij}^{(1)},i=1,2,j=1,2,3,4,$ are all independent λ . Hence we get

$$\Pi_{n+1}^{(N)}U_n=M_n\Pi_n^{(N)}. \quad (24)$$

Equating the coefficients of $\lambda^{N+i},i=1,2,$ in Eq. (24). we have

$$m_{11}^{(1)}=1,m_{11}^{(0)}=0,m_{12}^{(1)}=r_n-b_n^{(N-1)}=\widetilde{r}_n,m_{12}^{(0)}=0,m_{13}^{(1)}=0,m_{13}^{(0)}=0,m_{14}^{(1)}=u_n+b_n^{(N-1)}-g_n^{(N-1)},$$
$$m_{14}^{(0)}=0,m_{21}^{(0)}=s_n+c_{n+1}^{(N-1)}=\widetilde{s}_n,m_{22}^{(0)}=1+(r_n-b_n^{(N-1)})(s_n+c_{n+1}^{(N-1)})=1+\widetilde{r}_n\widetilde{s}_n,m_{23}^{(0)}=w_n-c_{n+1}^{(N-1)}$$
$$h_{n+1}^{(N-1)}=\widetilde{w}_n,m_{24}^{(0)}=(r_n-b_n^{(N-1)})(w_n-c_{n+1}^{(N-1)}+h_{n+1}^{(N-1)})+(s_n^{(N-1)}+c_{n+1}^{(N-1)})(u_n+b_n^{(N-1)}-f_n^{(N-1)})=\widetilde{r}_n\widetilde{w}_n+\widetilde{s}_n\widetilde{u}_n.$$

The proof is completed.

Proposition 2. The matrix $\widetilde{V}_n$ defined by (13) has the same form as V_n in Eq. (7) under the transformation (20), i.e,

$$\widetilde{V}_n=$$

$$\begin{pmatrix}\frac{\lambda}{2}-\widetilde{r}_n\widetilde{s}_{n-1} & \lambda\widetilde{r}_n & \frac{\lambda}{2}-\widetilde{r}_n\widetilde{s}_{n-1}-\widetilde{r}_n\widetilde{w}_{n-1}-\widetilde{s}_{n-1}\widetilde{u}_n & -\widetilde{r}_n\lambda+\widetilde{u}_n\lambda\\ \widetilde{s}_{n-1} & -\frac{\lambda}{2}+\widetilde{r}_n\widetilde{s}_{n-1} & -\widetilde{s}_{n-1}+\widetilde{w}_{n-1} & -\frac{\lambda}{2}+\widetilde{r}_n\widetilde{s}_{n-1}+\widetilde{r}_n\widetilde{w}_{n-1}+\widetilde{s}_{n-1}\widetilde{u}_n\\ 0 & 0 & \frac{\lambda}{2}-\widetilde{r}_n\widetilde{s}_{n-1} & \lambda\widetilde{r}_n\\ 0 & 0 & \widetilde{s}_{n-1} & -\frac{\lambda}{2}+\widetilde{r}_n\widetilde{s}_{n-1}\end{pmatrix}. \quad (25)$$

Proof: Let

$$(\Pi_n^{(N)})_t + \Pi_n^{(N)} V_n)(\Pi_n^{(N)})^* = \begin{pmatrix} \Sigma_{11}(\lambda,n) & \Sigma_{12}(\lambda,n) & \Sigma_{13}(\lambda,n) & \Sigma_{14}(\lambda,n) \\ \Sigma_{21}(\lambda,n) & \Sigma_{22}(\lambda,n) & \Sigma_{23}(\lambda,n) & \Sigma_{24}(\lambda,n) \\ 0 & 0 & \Sigma_{11}(\lambda,n) & \Sigma_{12}(\lambda,n) \\ 0 & 0 & \Sigma_{21}(\lambda,n) & \Sigma_{22}(\lambda,n) \end{pmatrix}.$$

We may get that $\Sigma_{11}(\lambda,n),\Sigma_{12}(\lambda,n),\Sigma_{13}(\lambda,n),\Sigma_{14}(\lambda,n),\Sigma_{24}(\lambda,n)$ and $\Gamma_{24}(\lambda,n)$ are $(4N+1)$ order polynomials in λ, $\Sigma_{21}(\lambda,n)$ and $\Gamma_{23}(\lambda,n)$ are $4N$ order polynomials in λ. Due to Eq.(14)~Eq.(18), by a detailed analysis, we can get

$$\Sigma_{ij}(\lambda_k,n)=0, \quad i=1,2, j=1,2,3,4, k=1,2,\cdots 2N,$$

and the following equation is established

$$((\Pi_n^{(N)})_t + \Pi_n^{(N)} V_n)(\Pi_n^{(N)})^* = \det(\Pi_n^{(N)}) Q_n. \tag{27}$$

Here

$$Q_n = \begin{pmatrix} q_{11}^{(1)}\lambda + q_{11}^{(0)} & q_{12}^{(1)}\lambda + q_{12}^{(0)} & q_{13}^{(1)}\lambda + q_{13}^{(0)} & q_{14}^{(1)}\lambda + q_{14}^{(0)} \\ q_{21}^{(0)} & q_{22}^{(1)}\lambda + q_{22}^{(0)} & q_{23}^{(0)} & q_{14}^{(1)}\lambda + q_{24}^{(0)} \\ 0 & 0 & q_{11}^{(1)}\lambda + q_{11}^{(0)} & q_{12}^{(1)}\lambda + q_{12}^{(0)} \\ 0 & 0 & q_{21}^{(1)} & q_{22}^{(1)}\lambda + q_{22}^{(0)} \end{pmatrix},$$

in which $q_{11}^{(1)},q_{12}^{(1)},q_{13}^{(1)},q_{14}^{(1)},q_{22}^{(1)},q_{24}^{(1)},q_{ij}^{(0)}, i=1,2, j=1,2,3,4,$ are all independent of λ.
From Eq. (27), we obtain

$$(\Pi_n^{(N)})_t + \Pi_n^{(N)} V_n = Q_n . \Pi_n^{(N)}. \tag{28}$$

Comparing the coefficients of λ in Eq. (28), we have

$$q_{11}^{(1)} = \frac{1}{2}, q_{11}^{(0)} = -(r_n - b_n^{(N-1)})(s_{n-1} + c_n^{(N-1)}) = -\tilde{r}_n \tilde{s}_{n-1}, q_{12}^{(1)} = r_n - b_n^{(N-1)} = \tilde{r}_n, q_{12}^{(0)} = 0,$$

$$q_{13}^{(1)} = -\frac{1}{2}, q_{13}^{(0)} = (r_n - b_n^{(N-1)})(s_{n-1} - c_n^{(N-1)}) - (r_n - b_n^{(N-1)})(w_{n-1} - c_n^{(N-1)} + h_n^{(N-1)}) - (s_{n-1}$$

$$+ c_n^{(N-1)})(u_n + b_n^{(N-1)} - g_n^{(N-1)}) = \tilde{r}_n \tilde{s}_{n-1} - \tilde{r}_n \tilde{w}_{n-1} - \tilde{s}_{n-1}\tilde{u}_n, q_{14}^{(1)} = -(r_n - b_n^{(N-1)}) + (u_n + b_n^{(N-1)}$$

$$- g_n^{(N-1)}) = -\tilde{r}_n + \tilde{u}_n, q_{14}^{(0)} = 0, q_{21}^{(0)} = s_{n-1} + c_n^{(N-1)} = \tilde{s}_{n-1}, q_{22}^{(1)} = -\frac{\lambda}{2}, q_{22}^{(0)} = (r_n - b_n^{(N-1)})(s_{n-1}$$

$$+ c_n^{(N-1)}) = \tilde{r}_n \tilde{s}_{n-1}, q_{23}^{(0)} = -(s_{n-1} + c_n^{(N-1)}) + (w_{n-1} - c_n^{(N-1)} + h_n^{(N-1)}) = -\tilde{s}_{n-1} + \tilde{w}_{n-1}, q_{24}^{(1)} = \frac{1}{2},$$

$$q_{24}^{(0)} = -(r_n - b_n^{(N-1)})(s_{n-1} - c_n^{(N-1)}) + (r_n - b_n^{(N-1)})(w_{n-1} - c_n^{(N-1)} + h_n^{(N-1)}) + (s_{n-1} + c_n^{(N-1)})(u_n$$

$$+ b_n^{(N-1)} - g_n^{(N-1)}) = -\tilde{r}_n \tilde{s}_{n-1} + \tilde{r}_n \tilde{w}_{n-1} + \tilde{s}_{n-1}\tilde{u}_n.$$

The proof is completed.

Hence we conclude that the transformation (9) and (13) can change the Lax pair (6) and (7) into another Lax pair with same form. As usual, the gauge transformations (9) and (20):

$$(\varphi_n; r_n, s_n, u_n, w_n) \to (\tilde{\varphi}_n; \tilde{r}_n, \tilde{s}_n, \tilde{u}_n, \tilde{w}_n). \tag{29}$$

is a Darboux transformation of the Eq. (6) and Eq. (7), and Eq. (20) is so-called a *Bäcklund* transformation (BT) between new solution $(\tilde{\varphi}_n; \tilde{r}_n, \tilde{s}_n, \tilde{u}_n, \tilde{w}_n)^T$ and old solution $(\varphi_n; r_n, s_n, u_n, w_n)^T$ .In conclusion, according to the propositions 1 and 2, we have the following

Theorem 1. Each solution $(\varphi_n; r_n, s_n, u_n, w_n)^T$ of the Eq. (5) is mapped into its a new solution $(\tilde{\varphi}_n; \tilde{r}_n, \tilde{s}_n, \tilde{u}_n, \tilde{w}_n)^T$ under the transformation (20), where $b_n^{(N-1)}, c_n^{(N-1)}, g_n^{(N-1)}, h_n^{(N-1)}$ are given by $Eq(14) \sim Eq.(17)$.

Remark 1. Because the highest order polynomial in elements of matrix $\Pi_n^{(N)}$ is N order polynomial, Darboux transformation (29) is also called N-fold Darboux transformation [19], $\Pi_n^{(N)}$ is also known as a matrix of N-fold Darboux transformation.

3. Application of Darboux Transformation

In this section, we will apply Darboux transformation (29) to find an explicit solution of Eq. (5). First, we chose a seed solution (i.e., the simple special solution) $(\varphi_n; r_n, s_n, u_n, w_n)^T = (0,0,e^t,e^{-t})^T$ of Eq. (5), and consider 1-fold Darboux transformation of Eq. (6) and Eq. (7), namely when $N=1$.Substituting this solution into the Eq. (6) and Eq. (7), we can obtain the following Lax pair

$$E\phi_n=\begin{pmatrix}\lambda & 0 & 0 & \lambda e^t\\ 0 & 1 & e^{-t} & 0\\ 0 & 0 & \lambda & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\phi_n,\quad \phi_{n,t}=\begin{pmatrix}\frac{\lambda}{2} & 0 & -\frac{\lambda}{2} & \lambda e^t\\ 0 & -\frac{\lambda}{2} & e^{-t} & \frac{\lambda}{2}\\ 0 & 0 & \frac{\lambda}{2} & 0\\ 0 & 0 & 0 & \frac{\lambda}{2}\end{pmatrix}. \tag{30}$$

Solving above two equations, we have

$$\phi_n=\begin{pmatrix}\left(\frac{\lambda\exp(-(\lambda-1)t)}{1-\lambda}-\frac{t}{2}\lambda^{n+1}\right)\exp(\frac{\lambda t}{2})\\ \left(\frac{\lambda t}{2}+\frac{\lambda^n\exp((\lambda-1)t)}{\lambda-1}\right)\exp(-\frac{\lambda t}{2})\\ \lambda^n\exp(\frac{\lambda t}{2})\\ \exp(-\frac{\lambda t}{2})\end{pmatrix},\psi_n=\begin{pmatrix}\left(\frac{\lambda\exp(-(\lambda-1)t)}{\lambda-1}-\frac{t}{2}\lambda^{n+1}\right)\exp(\frac{\lambda t}{2})\\ \left(-\frac{\lambda t}{2}+\frac{\lambda^n\exp((\lambda-1)t)}{\lambda-1}\right)\exp(-\frac{\lambda t}{2})\\ \lambda^n\exp(\frac{\lambda t}{2})\\ -\exp(-\frac{\lambda t}{2})\end{pmatrix}.$$

$$\alpha_j[n]=\frac{\left(\lambda_j\exp(-\frac{\lambda_j t}{2})(1+\kappa_j)+\left(\frac{\lambda_j^n\exp(-(\lambda_j-1)t)}{\lambda_j-1}(1-\kappa_j)\right)\exp(-\lambda_j t)\right)}{\left(\frac{\lambda_j\exp(-(\lambda_j-1)t)}{1-\lambda_j}(1+\kappa_j)-\frac{t}{2}\lambda_j^{n+1}(1-\kappa_j)\right)},i=1,2.$$

$$\beta_j[n]=\frac{\lambda_j^n(1-\kappa_j)\exp(\frac{\lambda_j t}{2})}{\left(\frac{\lambda_j\exp(-(\lambda_j-1)t)}{1-\lambda_j}(1+\kappa_j)-\frac{t}{2}\lambda_j^{n+1}(1-\kappa_j)\right)\exp(\frac{\lambda_j t}{2})},\quad i=1,2.$$

$$\alpha_j[n]=\frac{\exp(\frac{-\lambda_j t}{2})(1+\kappa_j)}{\left(\frac{\lambda_j\exp(-(\lambda_j-1)t)}{1-\lambda_j}(1+\kappa_j)-\frac{t}{2}\lambda_j^{n+1}(1-\kappa_j)\right)\exp(\frac{\lambda_j t}{2})},\quad i=1,2.$$

Finally, by means of $\alpha_j[n],\beta_j[n],\gamma_j[n],j=1,2,$ and transformation (20), we obtain a 1-fold solution of Eq. (5):

$$\widetilde{r}_n=-b_n^{(0)}=\frac{(\lambda_1-\lambda_2)\beta_1[n]\beta_2[n]}{\lambda_1\beta_2[n]\gamma_1[n]-\lambda_2\beta_1[n]\gamma_2[n]},\quad \widetilde{s}_n=c_{n+1}^{(0)}=\frac{(\lambda_1-\lambda_2)\gamma_1[n+1]\gamma_2[n+1]}{\beta_2[n+1]\gamma_1[n+1]-\beta_1[n+1]\gamma_2[n+1]},$$

$$\widetilde{u}_n=e^t+b_n^{(0)}-g_n^{(0)}=e^t-\frac{(\lambda_1-\lambda_2)\beta_1[n]\beta_2[n]}{\lambda_1\beta_2[n]\gamma_1[n]-\lambda_2\beta_1[n]\gamma_2[n]}+\frac{-1}{(\lambda_1\beta_2[n]\gamma_1[n]-\lambda_2\beta_1[n]\gamma_2[n])^2}\times$$
$$(\lambda_1^2\alpha_1[n]\beta_1[n](\beta_2[n])^2-(\beta_2[n])^2\gamma_1[n]\lambda_1^2-\beta_1[n](\beta_2[n])^2\gamma_1[n]\lambda_1^2-\alpha_2[n](\beta_1[n])^2\beta_2[n]\lambda_1\lambda_2$$
$$-\alpha_1[n]\beta_1[n](\beta_2[n])^2\lambda_1\lambda_2+(\beta_2[n])^2\gamma_1[n]\lambda_1\lambda_2+\beta_1[n](\beta_2[n])^2\gamma_1[n]\lambda_1\lambda_2+(\beta_1[n])^2\gamma_2[n]\lambda_1\lambda_2$$
$$+(\beta_1[n])^2\beta_2[n]\gamma_2[n]\lambda_1\lambda_2+\alpha_2[n](\beta_1[n])^2\beta_2[n]\lambda_2^2-(\beta_1[n])^2\gamma_2[n]\lambda_2^2-(\beta_1[n])^2\beta_2[n]\gamma_2[n]\lambda_2^2).$$

$$\widetilde{w}_n=e^{-t}-c_{n+1}^{(0)}+h_{n+1}^{(0)}=e^{-t}-\frac{(\lambda_1-\lambda_2)\gamma_1[n+1]\gamma_2[n+1]}{\beta_2[n+1]\gamma_1[n+1]-\beta_1[n+1]\gamma_2[n+1]}+\frac{(\lambda_1-\lambda_2)}{(\beta_2[n+1]\gamma_1[n+1]-\beta_1[n+1]\gamma_2[n+1])^2}\times(\alpha_2[n+1]\beta_2[n+1](\gamma_1[n+1])^2$$
$$-(\gamma_1[n+1])^2\gamma_2[n+1]+\beta_2[n+1](\gamma_1[n+1])^2\gamma_2[n+1]-\alpha_1[n+1]\beta_1[n+1](\gamma_2[n+1])^2\lambda_1+\gamma_1[n+1](\gamma_2[n+1]^2-\beta_1[n+1]\gamma_1[n+1](\gamma_2[n+1])^2.$$

Starting from the obtained explicit solution, we apply the Darboux transformation (29) once again, and then other new solution of Eq. (5) is obtained. This process can be done continually. Therefore, we can obtain a lot of explicit solutions for the differential-difference equation (5).

4. Conclusions and Remarks

In this paper, we have introduced an integrable coupling of the known differential-difference equation, and presented its Lax pair. With the help of a gauge transformation of the corresponding Lax pairs, a N-fold Darboux transformation of Lax for the triangular integrable coupling has been established. As an application an explicit solution is given by the obtained Darboux transformation.

Furthermore, we may also research other interesting integrability problems for the obtained integrable coupling, for instance, the inverse scattering transformation, constructing complexiton solutions by the Casorati determinant, and so on.

In addition, it should be noted that the construction method of Darboux transformations of the integrable couplings of the continuous integrable systems is similar our method in Section 2, the corresponding Darboux transformations of the integrable couplings of the continuous integrable systems will be researched in our future work.

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