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A Third Runge Kutta method based on a linear combination of arithmetic mean, harmonic mean and geometric mean

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Abstract: We present a new third order Runge Kutta method based on linear combination of arithmetic mean, geometric mean and harmonic mean to solve a first order initial value problem. We also derive the local truncation error and show the stability region for the method. Moreover, we compare the new method with Runge Kutta method based on arithmetic mean, geometric mean and harmonic mean. The numerical results show that the performance of the new method is the same as known third order Runge-Kutta methods.

Keywords: Initial Value Problems, Runge Kutta Method, Arithmetic Mean, Harmonic Mean, Geometric Mean.

1. Introduction

Consider a first initial value problem (IVP), which can be written in the form

$$y'(t) = f(t, y(t)), \ \ y(t_0) = y_0. \tag{1}$$

The autonomous structure for (1) is [6,p.43]

$$y'(t) = f(y(t)), \ \ y(t_0) = y_0. \tag{2}$$

A classical third order of Runge Kutta method to solve (1) defines [3] as follows

$$y_{n+1} = y_n + \frac{h}{4}\left(k_1 + 2k_2 + k_3\right), \tag{3}$$

where

$$k_1 = f(t_n, y_n) \tag{4}$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \tag{5}$$

$$k_3 = f(t_n + h, y_n - hk_1 + hk_2) \tag{6}$$

Since the formula (3) can be written in the form

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2}\right) \tag{7}$$

then, the method (7) is also named as Runge Kutta method based on arithmetic mean. Wazwaz [8] modifies the formula by replaced an arithmetic mean with harmonic mean, that is

$$\begin{aligned} y_{n+1} &= y_n + h\left(\frac{k_1k_2}{k_1+k_2} + \frac{k_2k_3}{k_2+k_3}\right) \\ k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1\right) \\ k_3 &= f\left(t_n + \frac{2h}{3}, y_n - \frac{2h}{3}k_1 + \frac{4h}{3}k_2\right) \end{aligned} \tag{8}$$

Evans [3] also modifies the formula (7) by replaced an arithmetic mean with a geometric mean. He ends up with the Runge Kutta method based on a geometric mean as

$$y_{n+1} = y_n + \frac{h}{2}\left(\sqrt{k_1 + k_2} + \sqrt{k_2 + k_3}\right)$$

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1\right) \\ k_3 &= f\left(t_n + \frac{2h}{3}, y_n - \frac{h}{2}k_1 + \frac{7h}{6}k_2\right) \end{aligned} \tag{10}$$

Moreover, Ababneh [1] proposes a weighted Runge Kutta method based on contraharmonic mean, given by

$$\begin{aligned} y_{n+1} &= y_n + h\left(\frac{k_1^2 + k_2}{k_1 + k_2} + \frac{k_2 k_3}{k_2 + k_3}\right) \\ k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1\right) \\ k_3 &= f\left(t_n + \frac{2h}{3}, y_n - \frac{2h}{3}k_1 + \frac{4h}{3}k_2\right) \end{aligned}$$

Where $w_1 = \dfrac{1}{4}$ and $w_2 = \dfrac{3}{4}$

Our purpose that to construct a method based on a linear combination of arithmetic mean (AM), harmonic mean (HM) and geometric mean (GM) as introduced by Khattri [4] as follows

$$RM\left(k_1, k_2\right) = \frac{14AM\left(k_1, k_2\right) - HM\left(k_1, k_2\right) + 32GM\left(k_1, k_2\right)}{45} \tag{11}$$

The structure of this paper is that in the section 2 we derive the method and it's truncation error, then it is followed by stability analysis of the method and numerical experiments. At the end we give a brief conclusion.

2. Runge Kutta Method Based on a Linear Combination of Arithemtic Mean, Harmonic Mean and Geometric Mean

We consider the problem (2). Replacing the arithmetic mean in (7) by (11), we have

$$\begin{aligned} y_{n+1} = y_n + \frac{h}{90}\Bigg(& 7\left(k_1 + 2k_2 + k_3\right) - \left(\frac{2k_1 k_2}{k_1 + k_2}\right. \\ & \left. + \frac{2k_2 k_3}{k_2 + k_3}\right) + 32\left(\sqrt{k_1 + k_2} + \sqrt{k_2 + k_3}\right)\Bigg) \end{aligned} \tag{12}$$

where

$$k_1 = f\left(y_n\right) \tag{13}$$

$$k_2 = f\left(y_n + h a_1 k_1\right) \tag{14}$$

$$k_3 = f\left(y_n + h a_2 k_1 + h a_3 k_2\right) \tag{15}$$

We have to finnd the appropriate a_1, a_2 and a_3 in order to get the third order method. By expanding k_1, k_2 and k_3, in (13), (14) and (15) using Taylor series, we obtain

$$k_1 = f\left(y_n\right) = f \tag{16}$$

$$k_2 = f + a_1 f f_y h + \frac{1}{2} a_1^2 f^2 f_{yy} h^2 \tag{17}$$

$$k_3 = f + \left(a_2 + a_3\right) f f_y h + \left(\frac{1}{2}\left(a_2 + a_3\right)^2 f^2 f_{yy} + a_1 a_3 f f_y^2\right) h^2 \tag{18}$$

Then we subtitute (16), (17 and (18) into (12). At this step, we have to evaluate the polynomial in the rational and square root form. To avoid this difficulty, we use geometric series and binomial series [5, p.236] and obtain

$$\frac{2k_1 k_2}{k_1 + k_2} = f + \frac{1}{2} a_1 f f_y h + \left(\frac{1}{4} a_1^2 f^2 f_{yy} - \frac{1}{4} a_1^2 f f_y^2\right) h^2 \tag{19}$$

$$\begin{aligned} \frac{2k_2 k_3}{k_2 + k_3} = f &+ \left(\frac{1}{2} a_1 f f_y + \frac{1}{2} a_2 f f_y + \frac{1}{2} a_3 f f_y\right) h \\ &+ \Big(a_1 a_3 f f_y^2 + \frac{1}{4} a_1^2 f^2 f_{yy} - \frac{1}{4} a_3^2 f f_y^2 \\ &+ \frac{1}{4} a_3^2 f^2 f_{yy} + \frac{1}{4} a_2^2 f^2 f_{yy} \\ &+ \frac{1}{4} a_1 a_2 f f_y^2 - \frac{1}{2} a_2 a_3 f f_y^2 + \frac{1}{2} a_2 a_3 f^2 f_{yy} \\ &- \frac{1}{4} a_1^2 f f_y^2 + \frac{1}{4} a_2^2 f f_y^2\Big) h^2 \end{aligned} \tag{20}$$

$$\begin{aligned} \sqrt{k_1 k_2} = f + \frac{1}{2} a_1 f f_y h &+ \Big(\frac{1}{2} a_1^2 f^2 f_{yy} \\ &- \frac{1}{8} a_1^2 f f_y^2\Big) h^2 \end{aligned} \tag{21}$$

$$\begin{aligned} \sqrt{k_2 k_3} = f &+ \Big(\frac{1}{2} a_2 a_3 f^2 f_{yy} + \frac{3}{4} a_1 a_3 f f_y^2 \\ &- \frac{1}{4} a_2 a_3 f f_y^2 + \frac{1}{2} a_3 f f_y + \frac{1}{2} a_2 f f_y \\ &+ \frac{1}{2} a_1 f f_y\Big) h + \Big(-\frac{1}{8} a_3^2 f f_y^2 + \frac{1}{4} a_3^2 f^2 f_{yy} \\ &- \frac{1}{8} a_2^2 f f_y^2 + \frac{1}{4} a_1^2 f^2 f_{yy} + \frac{1}{4} a_2^2 f^2 f_{yy} \\ &+ \frac{1}{4} a_1 a_2 f f_y^2 - \frac{1}{8} a_1^2 f f_y^2\Big) h^2 \end{aligned} \tag{22}$$

Subtituting (16), (17), (18), (19), (20), (21) and (22) into (12), we have

$$
\begin{aligned}
y_{n+1} = y_n + fh + &\left(\frac{1}{4}a_3 ff_y + \frac{1}{4}a_2 ff_y + \frac{1}{2}a_1 ff_y\right)h^2 \\
&+\left(\frac{1}{8}a_3{}^2 f^2 f_{yy} + \frac{1}{4}a_1{}^2 f^2 f_{yy} + \frac{1}{8}a_2{}^2 f^2 f_{yy}\right. \\
&-\frac{1}{12}a_2 a_3 ff_y{}^2 + \frac{1}{12}a_1 a_2 ff_y{}^2 + \frac{1}{3}a_1 a_3 ff_y{}^2 \\
&+\frac{1}{4}a_2 a_3 f^2 f_{yy} - \frac{1}{12}a_3{}^2 ff_y{}^2 - \frac{1}{24}a_2{}^2 ff_y{}^2 \\
&\left.-\frac{1}{24}a_1{}^2 ff_y{}^2\right)h^3
\end{aligned}
\tag{23}
$$

Comparing (23) with third order of Taylor

$$y_{n+1} = y_n + hf + \frac{1}{2}h^2 ff_y + \frac{1}{6}h^3\left(ff_y{}^2 + f^2 f_{yy}\right)$$

We obtain a nonlinear system of equation as follows

$$
\begin{aligned}
f^2 f_{yy} &: \frac{1}{4}a_2 a_3 + \frac{1}{4}a_1{}^2 + \frac{1}{8}a_2{}^2 + \frac{1}{8}a_3{}^2 = \frac{1}{6} \\
ff_y{}^2 &: \frac{1}{12}a_1 a_2 + \frac{1}{3}a_1 a_3 - \frac{1}{12}a_2 a_3 - \frac{1}{12}a_1{}^2 \\
&: -\frac{1}{24}a_2{}^2 - \frac{1}{24}a_3{}^2 = -\frac{1}{6} \\
ff_y &: \frac{1}{2}a_1 + \frac{1}{4}a_2 + \frac{1}{4}a_3 = \frac{1}{2}
\end{aligned}
\tag{24}
$$

Solving the system (24), we obtain $a_1 = \frac{2}{3}$, $a_2 = -\frac{4}{9}$ dan $a_3 = \frac{10}{9}$. By substituting the value of this parameter into equation (16), (17) and (18), we obtain the formula of Runge Kutta based on a linear combination of means as follows

$$
\begin{aligned}
y_{n+1} = y_n + \frac{h}{90}\Bigg(7\left(k_1 + 2k_2 + k_3\right) - &\left(\frac{2k_1 k_2}{k_1 + k_2}\right. \\
&\left.+\frac{2k_2 k_3}{k_2 + k_3}\right) + 32\left(\sqrt{k_1 + k_2} + \sqrt{k_2 + k_3}\right)\Bigg)
\end{aligned}
\tag{25}
$$

with

$$k_1 = f\left(t_n, y_n\right) \tag{26}$$

$$k_2 = f\left(t_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1\right) \tag{27}$$

$$k_3 = f\left(t_n + \frac{2h}{3}, y_n - \frac{4h}{9}k_1 + \frac{10h}{9}k_2\right) \tag{28}$$

The local truncation error of the method, can be obtained by expanding k_1, k_2 and k_3 in equation (26), (27) and (28) into Taylor series, then we substitute the resulting equatios in (25), we have

$$
\begin{aligned}
y_{n+1} = y_n + fh + \frac{1}{2}ff_y h^2 &+ \left(\frac{1}{6}f_y{}^2 + \frac{1}{6}f^2 f_{yy}\right)h^3 \\
&+\left(\frac{1}{162}ff_y{}^3 ++\frac{14}{81}f^2 f_y f_{yy} + \frac{1}{27}f^3 f_{yyy}\right)h^4
\end{aligned}
\tag{29}
$$

The local truncation error is given by the difference between fourth order of Taylor polynomial and equation (29), we get

$$LTE = h^4\left(-\frac{23}{648}f\, f_y{}^3 + \frac{1}{162}f^2 f_y f_{yy} - \frac{1}{216}f^3 f_{yyy}\right) \tag{30}$$

2. Stability Analysis

The stability region depends on the IVP. We can obtained this area by evaluate a differential equation $y' = \lambda y$, as the test of equation as suggested in [2, p.374]. Subtituting y' to (26), (27) and (28), we obtain

$$
\begin{aligned}
k_1 &= \lambda y \\
k_2 &= \lambda y + \frac{2}{3}h\lambda^2 y \\
k_3 &= \lambda y + \frac{2}{3}h\lambda^2 y + \frac{20}{27}h\lambda^3 y
\end{aligned}
\tag{31}
$$

Substituting (31) into (25), letting $z = \lambda h$ and applying binomial and geometric series, we obtain after simplifying

$$\frac{y_{n+1}}{y} = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{162}z^4 + \frac{1}{3645}z^5 . \tag{32}$$

Ignoring the terms in the right hand side of equation (32) for z^j, $j > 3$, we obtain for the stability equation for the method as follows

$$\frac{y_{n+1}}{y} = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 \tag{33}$$

or

$$\frac{y_{n+1}}{y} = 1 + \lambda h + \frac{1}{2}(\lambda h)^2 + \frac{1}{6}(\lambda h)^3 \tag{34}$$

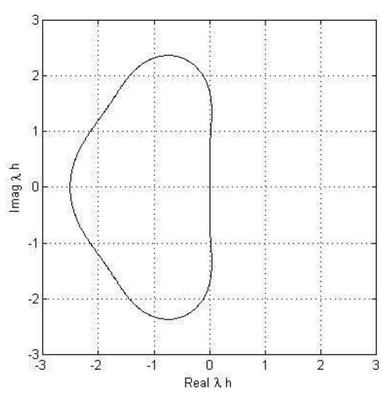

Figure 1. *Stability Region of the New Method*

Figure 1 shows the stability region of the method that is the set of complex values of λh for which all solutions of $y' = \lambda y$ will remain bounded as $n \to \infty$.

To highlight the performance of the new method, we use three the IVP's with various step- length, h ,as follows

1. $y' = \frac{1}{y}$, initial condition y(0)=1 and exact solution is $y = \sqrt{2x+1}$ on [0,1].
2. $y' = \frac{1}{1+x^2} - 2y^2$, initial condition y(0)=0 and exact solution is $y = \frac{1}{1+x^2}$ on [0,1].
3. $y' = y^2 (ln\, x)^3 - 2xy(ln\, x)^4 + 2ln\, x + 2$, initial condition y(1)=0 and exact solution is $y = 2xln\, x$ on [1,2].

To simplify the notations, we write the new method as RKMC, the method based on arithmetic mean as RKAM, the method based on harmonic mean as RKHM and the method based on geometric mean as RKGM. The computational results for three different step-length are given in Table 1. Instead using LTE in (30), we compute $LTE = \left\| y_i - y_{exact} \right\|_\infty$. From Table 1, we show that the proposed method is comparable with existing Runge Kutta third order method.

Table 1. *The Local Truncation Error for the Each Method With Various Length of h*

IVP	Length of *h*	RKMC	RKAM	RKHM	RKGM
Example 1	$h = 0.2$	6.570477e-005	1.031370e-003	1.059852e-005	5.244221e-005
	$h = 0.1$	7.490682e-006	2.488179e-004	1.600790e-006	6.072553e-006
	$h = 1/17$	1.470663e-006	8.489640e-005	3.483664e-007	1.194052e-006
Example 2	$h = 1/6$	1.327384e-003	2.274691e-003	4.151061e-003	2.096844e-003
	$h = 1/9$	5.090414e-004	8.722167e-004	1.571209e-003	7.932496e-004
	$h = 1/12$	2.790819e-004	4.519831e-004	8.481379e-004	4.303887e-004
Example 3	$h = 1/6$	4.428693e-004	3.189652e-004	9.314928e-004	6.240351e-004
	$h = 1/8$	2.376261e-004	1.877650e-004	6.438546e-004	3.392195e-004
	$h = 1/10$	1.474604e-004	1.255456e-004	4.068283e-004	2.123173e-004

4. Conclusions

We have shown how to construct a new formula for Runge Kutta method which is different from existing method. However the stability region of the new method is the same as other existing third order methods ([3], [7], [8]). Generally, the computational experiments show that the new method gives the similar results with the other methods. The local truncation error for the new method in many case, better than RKAM and give the almost same error with RKHM and RKGM. Hence, the new method is suitable for studying first order of Initial Value Problems.

References

[1] O.Y. Ababneh and R. Rozita, *New Third Order Runge Kutta Method Based on Contraharmonic Mean for Stiff Problems,* Applied Mathematical Sciences, 3(2009), 365-376.

[2] G. Dahlquist and A. Bjorck, Numerical Method, Prentice-Hall,Inc., New York, 1974.

[3] D.J. Evans, 1989. New Runge-Kutta Methods For Initial Value Problems, Applied Mathematics Letter, 2(1989), pp.25--28.

[4] S.K. Khattri, "Euler's Number and Some Means", Tamsui Oxford Journal of Information and Mathematical Sciences, 28(2012), pp.369--377.

[5] K.A. Ross, Elementary Analysis, Springer, New York, 1980.

[6] L. P. Shampine, *Numerical of Ordinary Diferential Equation*, Chapman and Hall, New York, 1994.

[7] A.G. Ujagbe, "On the Stability Analysis of a Geometric Mean 4th Order Runge-Kutta Formula, Mathematical Theory and Modelling, 3(2013), pp.76-91.

[8] A.M. Wazwaz, "A Modified Third Order Runge-Kutta Method", Applied Mathematics Letter, 3(1990), pp.123-125.