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Zeros and asymptotic limits of Löwdin orthogonal polynomials with a unified view

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Abstract: The zeros and asymptotic limits of two new classes of orthogonal polynomials, which are derived by applying two orthogonalization procedures due to Löwdin to a set of monomials, are calculated. It is established that they possess all the properties ofthe zeros of a polynomial. Their asymptotic limits are found. A Unified view of all the Löwdin orthogonal polynomials together with the standard classical orthogonal polynomials are presented in a unique graph.

Keywords: Asymptotic Limits, Canonical Orthogonalization, Complex Zeros, Hermitian Metric Matrix, Positive-Definiteness, Symmetric Orthogonalization

1. Introduction

Various methods of obtaining orthogonal polynomials, its zeros, asymptotic limits and their applications to various branches have been in regular progress. The classical orthogonal polynomials such as the Legendre, Hermite, Laguerre etc. obtained from a set of monomials/functions $\{x^N\}$, $N = 0, 1, 2, \dots, \infty$, are orthogonal and orthonormal in a given interval [a, b] with respect to a weight function $w(x)$. The Gram-Schmidt orthogonalization method [1] sequentially generates them upto any order without having any restriction on the order of the polynomials. Hence they do not have any asymptotic limits on their generation.

We have extended the application of the Löwdin orthogonalization procedures to a set of monomials $\{x^N\}$, $N = 0, 1, 2, \dots, \infty$, for the classical orthogonal polynomials such as the Chebyshev I, Chebyshev II, Gegenbauer, Jacobi and Bessel polynomials with their respective weight functions and limits of integration.

The repetition of this procedure with Löwdin's symmetric and canonical orthogonalizations [2, 3, 4] have restrictions on the generation of these two new classes of orthogonal polynomials. Though their properties such as zeros and asymptotic limits satisfy most of the properties of a polynomial, yet the Löwdin orthogonalization methods have a restriction on the generation of new polynomials because of the positive-definiteness of the Hermitian metric matrix formed from the inner product of two consecutive monomials for higher values of N.

2. Classical Orthogonal Polynomials

The classical orthogonal polynomials such as the Legendre, Hermite, Laguerre etc. are obtained from a set of monomials/functions $\{x^N\}$, $N = 0, 1, 2, \dots, \infty$, which are orthogonal and orthonormal in a given interval [a, b] with respect to a weight function $w(x)$. They can be generated one by one using the Gram-Schmidt orthogonalization procedure using a given set of monomials/functions. These set of monomials/functions can be taken one at a time in an increasing order. The polynomials obtained are a direct consequence of the choice of particular interval and weight function. The use of monomials with different choice of intervals and weight functions leads to different sets of orthogonal polynomials. In fact, these polynomials are solutions of particular differential equations. The choice of interval, weight function, orthogonality and orthonormalization conditions for different orthogonal polynomials are listed in Table 1.

Table 1. *List of the choice of interval, weight function, orthogonality and Orthonormalization conditions for different classical orthogonal polynomials*

Sl. No.	Name of the Polynomial	Interval [a, b]	Weight Function $w(x)$	Orthogonality Condition	Orthognormality Condition
1.	Legendre	[-1, 1]	1	$\int_{-1}^{1} P_m(x)P_n(x)dx$	$\int_{-1}^{1} P_n{}^2(x)dx = 1$
2.	Hermite	[-∞,∞]	e^{-x^2}	$\int_{-\infty}^{\infty} H_m(x)H(x)dx$	$\int_{-\infty}^{\infty} H_n{}^2(x)dx = 1$
3.	Laguerre	[0, ∞]	e^{-x}	$\int_{0}^{\infty} L_m(x)L_n(x)dx$	$\int_{0}^{\infty} L_n{}^2(x)dx = 1$
4.	Chebyshev I	[-1, 1]	$(1-x^2)^{-1/2}$	$\int_{-1}^{1} C_m(x)C_n(x)dx$	$\int_{-1}^{1} C_n{}^2(x)dx = 1$
5.	Chebyshev I	[0, -1]	$(1-x^2)^{1/2}$	$\int_{-1}^{1} C_m(x)C_n(x)dx$	$\int_{-1}^{1} C_n{}^2(x)dx = 1$
6.	Gegenbauer	[-1, 1]	$(1-x^2)^{\alpha-1/2}$	$\int_{-1}^{1} G_m(x)G_n(x)dx$	$\int_{-1}^{1} G_n{}^2(x)dx = 1$
7.	Bessel	[0, ∞]	x	$\int_{-1}^{1} B_m(x)B_n(x)dx$	$\int_{0}^{\infty} B_n{}^2(x)dx = 1$
8.	Jacobi	[-1, 1]	$(1-x)^{\alpha}(1+x)^{\beta}$	$\int_{0}^{\infty} J_m(x)J_n(x)dx$	$\int_{-1}^{1} J_n{}^2(x)dx = 1$

3. Löwdin Methods for Functions

The Gram-Schmidt procedure sequentially orthogonalizes a given set of linearly independent monomials. The two methods due to Löwdin are democratic in the sense that they handle all the given monomials simultaneously and treat them on equal footing. The derivation of the two procedures is described in detail in references [2, 4]. However, we have briefly given their derivation here for completeness and to highlight the interesting properties they possesses.

Consider a set of linearly independent monomials/functions $V = \{x^N\}$, $N = 0, 1, 2, \dots, \infty$. We can define a general non-singular linear transformation A for the basis V to go to a new basis Z:

$$Z = VA \tag{1}$$

The set $Z(\equiv \{z_k\})$ will be orthonormal if

$$< Z|Z > = < VA|VA > = A < V|V > A = AMA = I \tag{2}$$

where M is a Hermitian metric matrix of the given basis V.

For any consecutive monomials $f(x)$ and $g(x)$ of V, the Hermitian metric matrix can be constructed using

$$M = M_{ij} = \int_a^b w(x)f^*(x)g(x)dx \tag{3}$$

4. Positive Semi-Definite Matrix and Asymptotic Limits

The new classes of orthogonal polynomials [7] are obtained by constructing the Hermitian metric matrix from the given set of monomials and then finding its eigenvalues and eigenvectors. Now using the computed eigenvectors U of M and $Ud^{-1/2}$, the symmetric and canonical orthogonal polynomials are constructed by multiplying them respectively by the set of monomials x^N's. As the order N of the monomials increases, the Gram matrix becomes positive semi-definite which results in getting some of its eigenvalues negative which in turn produces complex eigenvectors. This puts a serious limit on finding the new sets of Löwdin orthogonal polynomials for higher values of N. We have thoroughly checked the asymptotic limits of all the new classes of Löwdin orthogonal polynomials by increasing the order of the monomials one by one.

The asymptotic limits in the case of Löwdin-Legendre, Löwdin-Chebyshev type I and type II polynomials is $N = 8$; for Löwdin-Hermite polynomials, it is 13; for Löwdin-Jacobi polynomials, it is 10 and for Löwdin-Bessel polynomials, it is 6. Because of the positive semi-definite nature of the Gram matrix, the Löwdin-Bessel and Löwdin-Laguerre polynomials begins to show signs of not being so accurately orthogonal as early as $N = 3$.

5. Zeros and Their Properties

We have computed the zeros of all the Löwdin orthogonal polynomials and they are listed in Tables from 2 to 8. The zeros are both real and complex for both the cases of two new sets of Löwdin polynomials. The zeros of these two new families of orthogonal polynomials satisfy all the properties possessed by the zeros of classical orthogonal polynomials. These zeros are real and distinct in the interval [a, b] and are located in the interior of the interval. Each interval comprises precisely unique root only. There are cases where the zeros are imaginary but is of incisively single root. The real or imaginary root lies either in the interval [a, b] or in the exterior of [a, b].

Table 2. *Zeros of Löwdin-Legendre Orthogonal Polynomials*

Löwdin-Legendre	**Löwdin-Legendre**
Symmetric Polynomials	Canonical Polynomials
N = 2:	**N = 2:**
1.2224; -1.2224	0 + 1.6618 *i*; 0 -0.6618 *i*
0	0
0.5067; -0:5067	0.6017; -0.6017
N = 3:	**N = 3:**
1.2224; -1.2224	0 + 1.6618 *i*; 0-0.6618 *i*
0; 1.0727; -1.0727	0; 0 + 1.2585 *i*; 0 – 1.2585 *i*
0.5067; - 0.5067	0.6017; - 0.6017
0; 0.6895; - 0.6895	0; 0.7946; - 0.7946
N = 4:	**N = 4:**
- 1.2632 + 0.3623 *i*; - 1.2632 + 0.3623 *i*; 1.2632 + 0.3623 *i*; 1.2632 0.3623 *i*	- 0.7925 + 1.1907 *i*; - 0.7925 + 1.1907 *i*; 0.7925 + 1.1907 *i*; 0.7925 + 1.1907 *i*
0; 1.0727; - 1.0727	0; 0 + 1.2585 *i*; 0 - 1.2585 *i*
1.0246; - 1.0246; 0.4050; - 0.4050	0 + 1.2147 *i*; 0 - 1.2174 *i*; 0.6553; - 0.6553
0; 0.6895; 0.6895	0.0.7946; - 0.7946
0.7880; - 0.7880; 0.2548; - 0.2548	0.8739; - 0.8739; 0.3571; - 0.3571

Table 3. *Zeros of Löwdin-Hermite Orthogonal Polynomials*

Löwdin-Hermite	**Löwdin-Hermite**
Symmetric Polynomials	Canonical Polynomials
N = 2:	**N = 2:**
1.7070; -1.7070	0 + 1.1318 *i*; 0-0.1318 *i*
0	0
0.5412; - 0.5412	0.8836; -0.8836
N = 3:	**N = 3:**
1.7070; -1.7070	0 + 1.1318 *i*; 0 - 0.1318 *i*
0; 1.8210; -1.8210	0; 0 + 0.6633 *i*; 0 - 0.6633 *i*
0.5412; - 0.5412	0.8836; -0.8836
0; 0.8212; -0.8212	0; 1.5077; -1.5077
N = 4:	**N = 4:**
2.5712; -2.5712;	-0.3391 + 0.5150 *i*; -0.3391- 0.5150 *i*;
1.6493; - 1.6493	0.3391 + 0.5150 *i*; 0.3391- 0.5150 *i*
0; 1.8210; - 1.8210	0; 0 + 0.6633 *i*; 0- 0.6633 *i*
2.0486; -2.0486; 0.4980; -0.4980	1.6727; -1.6727; 0 + 1.2408 *i*; 0- 1.2408 *i*
0; 0.8212; - 0.8212	0.1.5077; - 1.5077
0.9935; - 0.9935; 0.3546; - 0.3546	1.9016; - 1.9016; 0.6225; - 0.6255

Table 4. *Zeros of Löwdin-Laguerre Orthogonal Polynomials*

Löwdin-Laguerre	**Löwdin-Laguerre**
Symmetric Polynomials	Canonical Polynomials
N = 2:	**N = 2:**
6.3403; 2.5447	-0.1283 + 0.2739 *i*; -0.1283- 0.2739 *i*
4.6922; 0.4489	3.5984; -1.0616
1.0590; 0.3054	4.3666; 0.8539
N = 3:	**N = 3:**
8.9828; 5.0553; 2.4933	-0.2045; 0.0180 + 0.2005 *i*; 0.0180 - 0.2005 *i*
8.1509; 3.5793; 0.3981	4.7558; 0.2666 + 0.5458 *i*; 0.2666 - 0.5458 *i*
6.7333; 1.1750; 0.2451	6.6737; 2.1710; -1.1358
1.0398 + 0.6141 *i*; 1.0398 - 0.6141 *i*; 0.1819	7.4195; 2.9634; 0.5803

Table 5. *Zeros of Löwdin-Chebyshev I Orthogonal Polynomials*

Löwdin-Chebyshev I	**Löwdin-Chebyshev I**
Symmetric Polynomials	Canonical Polynomials
N = 2:	**N = 2:**
1.2069; -1.2069	0 + 1.3431 *i*; 0 - 1.3431 *i*
0	0
0.6079; -0.6079	0.7446; -0.7446
N = 3:	**N = 3:**
1.2069; -1.2069	0 + 1.3431 *i*; 0 - 1.3431 *i*
0; 1.0801; -1.0801	0; 0 + 1.1317 *i*; 0 - 1.1317 *i*
0.6079; -0.6079	0.7446; -0.7446
0; 0.7746; -0.7746	0; 0.8836; -0.8836
N = 4:	**N = 4:**
1.3141 + 0.2664 *i*; 1.3141 - 0.2664 *i*; -1.3141 + 0.2664 *i*; -1.3141 - 0.2664 *i*	0.6448 + 1.0232 *i*; 0.6448 - 1.0232 *i*; -0.6448 + 1.0232 *i*; -0.6448 - 1.0232 *i*
0; 1.0801;-1.0801	0; 0 + 1.1317 *i*; 0 - 1.1317 *i*
1.0379; -1.0379; 0.4522;-0.4522	0 + 1.2176 *i*; 0- 1.2176 *i*; 0.8048; -0.8048
0; 0.7746; -0.7746	0; 0.8836; -0.8836
0.8579; - 0.8579; 0.2704; -0.2704	0.9327; - 0.9327; 0.4048; -0.4048

Table 6. *Zeros of Löwdin-Chebyshev II Orthogonal Polynomials*

Löwdin-Chebyshev II	**Löwdin-Chebyshev II**
Symmetric Polynomials	Canonical Polynomials
N = 2:	**N = 2:**
1.2237; - 1.2237	0 + 1.9403 *i*; 0 -s 1.9403 *i*
0	0
0.4473; - 0.4473	0.5154; - 0.5154
N = 3:	**N = 3:**
1.2237; - 1.2237	0 + 1.9403 *i*; 0 - 1.9403 *i*
0; 1.0622; - 1.0622	0; 0 + 1.3791 *i*; 0 - 1.3791 *i*
0.4473; - 0.4473	0.5154; - 0.5154
0; 0.6341; - 0.6341	0; 0.7251; - 0.7251
N = 4:	**N = 4:**
1.2254 + 0.3923 *i*; 1.2254 - 0.3923 *i*; -1.2254 + 0.3923 *i*; -1.2254 - 0.3923 *i*	0.9213 + 1.3444 *i*; 0.9213 - 1.3444 *i*; -0.9213 + 1.3444 *i*; -0.9213 - 1.3444 *i*
0; 1.0622; -1.0622	0; 0 + 1.3791 *i*; 0 - 1.3791 *i*
1.0101; -1.0101; 0.3721; -0.3721	0 + 1.2428 *i*; 0 - 1.2428 *i*; 0.5561; -0.5561
0; 0.6314; -0.6314	0.0.7251; -0.7251
0.7367; - 0.7367; 0.2409; - 0.2409	0.8232; - 0.8232; 0.3232; - 0.3232

Table 7. *Zeros of Löwdin-Bessel Orthogonal Polynomials*

Löwdin-Besesl	**Löwdin-Bessel**
Symmetric Polynomials	Canonical Polynomials
N =2:	**N = 2:**
1.1524 + 0.3810 *i*;	-0.6414 + 1.1812 *i*;
1.1524 - 0.3810 *i*	-0.6414 - 1.1812 *i*
1.0109; 0.3891	-1.3047; 0.7346
0.7385; 0.2220	0.8546; 0.3691
N =3:	**N = 3:**
1.1041;0.6261+0.6963 *i*;	-1.2925; 0.0420 + 1.2805 *i*;
0.6261 – 0.6963 *i*	0.0420 - 1.2805 *i*
0.8845 + 0.1484 *i*;	-0.7969 + 0.9625 *i*;
0.8845 – 0.1484 *i*;0.3003	-0.7969 - 0.9625 *i*; 0.7638
0.9699; 0.6040; 0.1809	-1.2223; 0.8743; 04141
0.8463; 0.4863; 0.1635	0.9188; 0.6087; 0.2079

Table 8. *Zeros of Löwdin-Jacobi Orthogonal Polynomials*

Löwdin-Jacobi	**Löwdin-Jacobi**
Symmetric Polynomials	Canonical Polynomials
N = 2:	**N = 2:**
1.2234; -1.2234	0 + 2.1867 *i*; 0 - 2.1867 *i*
0	0
0.4059; -0.4059	0.4574; -0.4574
N = 3:	**N = 3:**
1.2234; -1.2234	0 + 2.1867 *i*; 0 - 2.1867 *i*
0; 1.0497; -1.0497	0.4574; - 0.4574
0; 0.5892; - 0.5892	0; 0.6700; - 0.6700
N = 4:	**N = 4:**
1.1933 + 0.4410 *i*;	1.0332 + 1.4819 *i*;
1.1933 - 0.4410 *i*;	1.0332 - 1.4819 *i*;
-1.1933 + 0.4410 *i*;	-1.0332 + 1.4819 *i*;
-1.1933 - 0.4410 *i*	-1.0332 - 1.4819 *i*
0; 1.0497; - 1.0497	0; 0 + 1.4925 *i*; 0 - 1.4925 *i*
0.9917; - 0.9917;	0 + 1.2832 *i*; 0 - 1.2832 *i*;
0.3459; -0.3459	0.4882; - 0.4882
0; 0.5892; - 0.5892	0.0.6700; - 0.6700
0.6796; - 0.6796;	0.7792; - 0.7792;
0.2304; - 0.2304	0.2967; - 0.2967

Table 9. *Löwdin-Legendre Symmetric and Canonical Coefficients*

Löwdin-Legendre	**Löwdin-Legendre**
Symmetric Orthogonalization	Canonical Orthogonalization
N = 2:	**N = 2:**
$s_0 = 1.0782; s_1 = 0; s_2 = -0.3933$	$c_0 = 0.7038; c_1 = 0; c_2 = 0.1516$
$s_0 = 0; s_1 = 1.2247; s_2 = 0$	$c_0 = 0; c_1 = 1.2247; c_2 = 0$
$s_0 = 0.1759; s_1 = 0; s_2 = 1.5315$	$c_0 = - 0.0678; c_1 = 0; c_2 = 1.5739$
N = 3:	**N = 3:**
$s_0 = 1.0782; s_1 = 0;$	$c_0 = 0.7038; c_1 = 0;$
$s_2 = - 0.3933; s_3 = 0$	$c_2 = 0.1516; c_3 = 0$
$s_0 = 0; s_1 = 1.1059;$	$c_0 = 0; c_1 = 1.2160;$
$s_2 = 0; s_3 = - 0.8034$	$c_2 = 0; c_3 = 0.2227$
$s_0 = 0.1759; s_1 = 0;$	$c_0 = - 0.0678; c_1 = 0;$
$s_2 = 1.5315; s_3 = 0$	$c_2 = 1.5739; c_3 = 0$
$s_0 = 0; s_1 = 0.5260;$	$c_0 = 0; c_1 = - 0.1456;$
$s_2 = 0; s_3 = 1.6896$	$c_2 = 0; c_3 = 1.8576$
N = 4:	**N = 4:**
$s_0 = 0.6806; s_1 = 0; s_2 = -0.4261;$	$c_0 = 0.6993; c_1 = 0; c_2 = 0.2312;$
$s_3 = 0; s_4 = 0.0705$	$c_3 = 0; c_4 = 0.0325$
$s_0 = 0; s_1 = 1.1059; s_2 = 0;$	$c_0 = 0; c_1 = 1.2160; c_2 = 0;$
$s_3 = - 0.8034; s_4 = 0$	$c_3 = 0.2227; c_4 = 0$
$s_0 = 0.1700; s_1 = 0; s_2 = 1.2476;$	$c_0 = - 0.1041; c_1 = 0; c_2 = 1.5502;$
$s_3 = 0; s_4 = - 1.1994$	$c_3 = 0; c_4 = 0.2783$
$s_0 = 0; s_1 = 0.5260; s_2 = 0;$	$c_0 = 0; c_1 = - 0.1456; c_2 = 0;$
$s_3 = 1.6896; s_4 = 0$	$c_3 = 1.8576; c_4 = 0$
$s_0 = 0.0893; s_1 = 0; s_2 = 0.8728;$	$c_0 = 0.0031; c_1 = 0; c_2 = - 0.2086;$
$s_3 = 0; s_4 = 1.7485$	$c_3 = 0; c_4 = 2.1030$

Table 10. *Löwdin-Hermite Symmetric and Canonical Coefficients*

Löwdin-Hermite	**Löwdin-Hermite**
Symmetric Orthogonalization	Canonical Orthogonalization
N = 2:	**N = 2:**
$s_0 = 0.7207; s_1 = 0; s_2 = - 0.0747$	$c_0 = 0.6981; c_1 = 0; c_2 = 0.0980$
$s_0 = 0; s_1 = 0.5311; s_2 = 0$	$c_0 = 0; c_1 = 0.5311; c_2 = 0$
$s_0 = 0.2112; s_1 = 0; s_2 = 0.2549$	$c_0 = \quad 0.2773; c_1 = 0; c_2 = 0.2468$
N = 3:	**N = 3:**
$s_0 = 0.7207; s_1 = 0;$	$c_0 = 0; c_1 = 0.4493;$
$s_2 = -0.0747; s_3 = 0$	$c_2 = 0; c_3 = 0.0579$
$s_0 = 0; s_1 = 0.4404;$	$c_0 = \quad 0.6981; c_1 = 0;$
$s_2 = 0; s_3 = - 0.0606$	$c_2 = - 0.098; c_3 = 0$
$s_0 = 0:2112; s_1 = 0;$	$c_0 = 0:2773c_1 = 0;$
$s_2 = 0:2549; s_3 = 0$	$c_2 = - 0:2468; c_3 = 0$
$s_0 = 0; s_1 = 0:2970;$	$c_0 = 0; c_1 = 0:2834;$
$s_2 = 0; s_3 = 0:0899$	$c_2 = 0; c_3 = - 0:0917$
N = 4:	**N = 4:**
$s_0 = 0:7136; s_1 = 0;$	$c_0 = 0:2773; c_1 = 0;$
$s_2 = \quad -0:0799;$	$c_2 = 0:2191;$
$s_3 = 0; s_4 = 0:0032$	$c_3 = 0; c_4 = 0:0166$
$s_0 = 0; s_1 = 0:4404; s_2 = 0;$	$c_0 = 0; c_1 = - 0:8985; c_2 = 0;$
$s_3 = - 0:0606; s_4 = 0$	$c_3 = - 0:0579; c_4 = 0$
$s_0 = 1:5595; s_1 = 0; s_2 = 0:8459;$	$c_0 = - 0:6932; c_1 = 0; c_2 = - 0:0727;$
$s_3 = 0; s_4 = - 0:0284$	$c_3 = 0; c_4 = - 0:0104$
$s_0 = 0; s_1 = 0:2970; s_2 = 0;$	$c_0 = 0; c_1 = 0:2834; c_2 = 0;$
$s_3 = 0:0899; s_4 = 0$	$c_3 = - 0:0917; c_4 = 0$
$s_0 = 0:1295; s_1 = 0; s_2 = 0:1924;$	$c_0 = - 0:0791; c_1 = 0; c_2 = 0:1322;$
$s_3 = 0; s_4 = 0:0255$	$c_3 = 0; c_4 = 2:1030$

Table 11. *Löwdin-Laguerre Symmetric and Canonical Coefficients*

Löwdin-Laguerre	**Löwdin-Laguree**
Symmetric Orthogonalization	Canonical Orthogonalization
N = 2:	**N = 2:**
$s_0 = 0:8685; s_1 = 0:4587;$	$c_0 = 0:1995; c_1 = -0:8096;$
$s_2 = 0:0939$	$c_2 = 0:1902$
$s_0 = 0:4185; s_1 = - 0:4700;$	$c_0 = - 0:8692; c_1 = - 0:2919;$
$s_2 = -0:3916$	$c_2 = 0:1995$
$s_0 = 0:2784; s_1 = - 0:7651;$	$c_0 = 0:2140; c_1 = 0:5071;$
$s_2 = 0:2903$	$c_2 = 0:4171$
N = 3:	**N = 3:**
$s_0 = 0:8652; s_1 = 0:4617;$	$c_0 = 0:2309; c_1 = - 0:6772;$
$s_2 = 0:1077; s_3 = 0:0143$	$c_2 = 0:3319; c_3 = - 0:0362$
$s_0 = 0:3933; s_1 = -0:3298;$	$c_0 = \quad 0:7961; c_1 = 0:5785;$
$s_2 = - 0:3097; s_3 = 0:0990$	$c_2 = 0:2542; c_3 = - 0:0685$
$s_0 = 0:2921; s_1 = -0:5807;$	$c_0 = 0:7336; c_1 = 0:5363;$
$s_2 = - 0:1032; s_3 = 0:1219$	$c_2 = 0:0825; c_3 = - 0:0639$
$s_0 = 0:1643; s_1 = - 0:5816;$	$c_0 = - 0:0896; c_1 = - 0:2872;$
$s_2 = 0:3632; s_3 = - 0:0539$	$c_2 = - 0:2611; c_3 = - 0:1330$

We have calculated the Löwdin's symmetric and canonical coefficients for the case of all the newly generated orthogonal polynomials using equation (10) of [7].They are listed in Tables 9 to 11.

6. Unified View

A Unified view of all the polynomials obtained using the three orthogonalization methods viz., the Gram-Schmidt, symmetric and canonical are presented in a unique graph as shown in Figures 1 to 7. This unification helps to visualize the three kinds of orthogonal polynomials at a time. We have used different symbols for different kind of polynomials. In this unified graphical representation, except Bessel and Laguerre orthogonal polynomials, big differences are there in other polynomials. The classical orthogonal polynomials obtained through Gram-Schmidt method are represented by solid lines (-), the Löwdin symmetric orthogonal polynomials are represented by dashed lines (--) and the Löwdin canonical orthogonal polynomials are represented by dotted lines (:::). A particular colour represent one kind of polynomial in each of the three polynomials. The different colours; magenta, green, red and blue represent the 0^{th}; 1^{st}; 2^{nd} and 3^{rd} polynomials respectively.

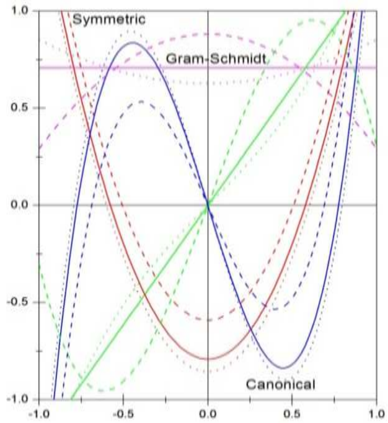

Figure 1. *Unified view of Legendre and Löwdin-Legendre Polynomials*

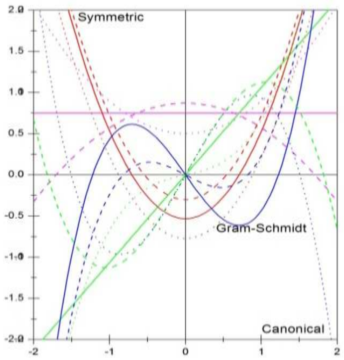

Figure 2. *Unified view of Hermite and Löwdin-Hermite Polynomials*

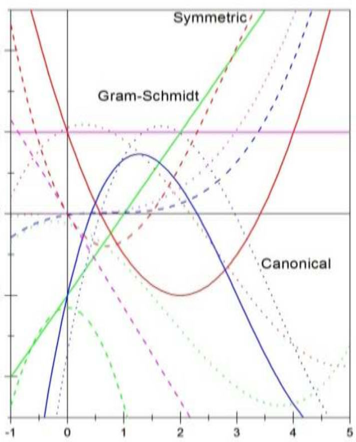

Figure 3. *Unified view of Laguerre and Löwdin-Laguerre Polynomials*

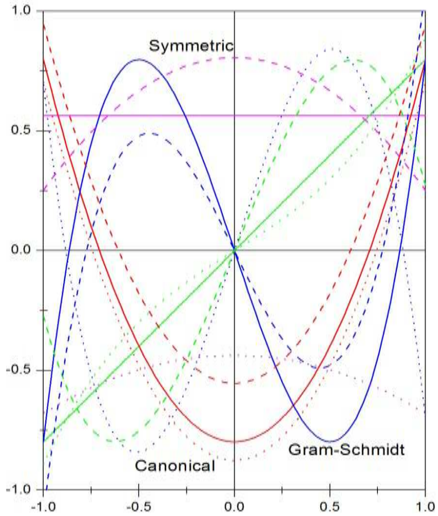

Figure 4. *Unified view of Chebyshev I and Löwdin-Chebyshev I Polynomials*

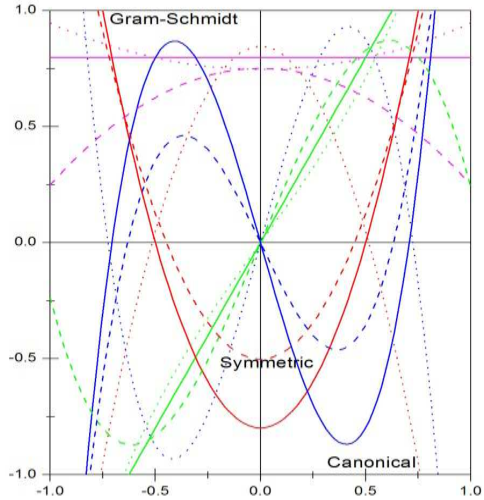

Figure 5. *Unified view of Chebyshev II and Löwdin-Chebyshev II Polynomials*

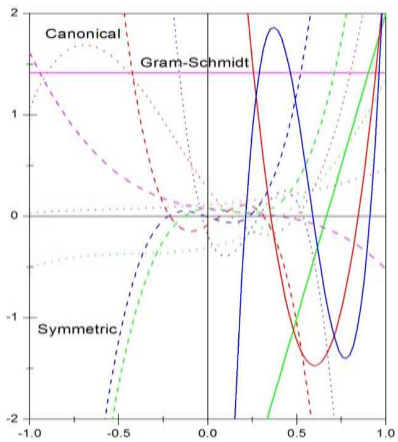

Figure 6. *Unified view of Bessel and Löwdin-Bessel Polynomials*

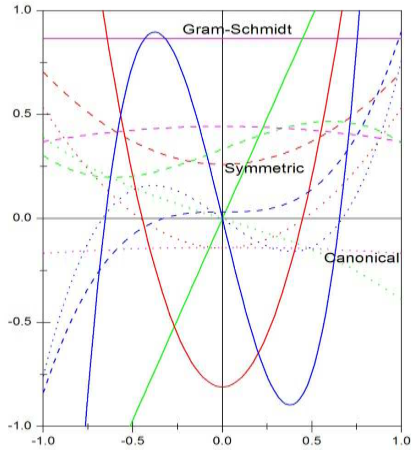

Figure 7. *Unified view of Jacobi and Löwdin-Jacobi Polynomials*

7. Conclusions

In sum, the general concept of orthogonal polynomials have been introduced describing many new classes of orthogonal polynomials. Two new sets of orthogonal polynomials can be easily obtained by applying the concept of Löwdin orthogonalization methods. The zeros and asymptotic limits of the Löwdin orthogonal polynomials are calculated. Quite understandably, the biggest difference arises in the Löwdin-Bessel and Löwdin-Laguerre polynomials when compared with their standard orthogonal polynomials. There are some shortcomings in finding the eigenvalues and eigenvectors of the Hermitian metric matrix M for higher values of$N$. An unfavorable feature of the Gram matrix M is that it begins to develop negative eigenvalues beyond a certain order N of the monomials.

Löwdin polynomials obtained through symmetric and canonical orthogonalization procedures are compared with the classical polynomials obtained through Gram-Schmidt orthogonalization procedure for $N = 2$; 3 and 4 (i.e. 3; 4 and 5 monomials). All the polynomials are normalized to *unity*. It is observed that alternate polynomials in a set repeat in the subsequent set.

All the newly obtained polynomials are plotted and are compared with the standard orthogonal polynomials obtained through the Gram-Schmidt orthogonalization procedure.

This theoretical framework offers the possibility to search for new iterative approaches where desired properties can be immediately obtained from the theoretical investigations. Moreover, a classification of the qualitative behaviour of these polynomials with a different view point has been introduced. The Löwdin orthogonalization concepts and these new classes of orthogonal polynomials could be the basis to develop new methods and new polynomials. From these results, research for efficient and robust techniques could be stimulated. With the new classes of orthogonal polynomials, the subject of orthogonal polynomials will witness the possible openings of several new avenues of research. They may open problems in numerical analysis, solutions of non-linear differential equations, least-square curve fitting etc.

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